JOURNAL OF THE AERONAUTICAL SCIENCES

VOLUME 22 JUNE, 1955 NUMBER 6

CONTENTS

Fatigue Life of Airplane Structures (Eighteenth Wright Brothers Lecture) ..... BO LUNDBERG 349

Separation of a Supersonic Turbulent Boundary Layer ..... S. M. BOGDONOFF AND C. E. KEPLER 414

Laminar Heat Transfer in Two-Dimensional Subsonic Effusors ..... PAUL A. LIBBY AND MARIAN VISICH, JR. 425

Readers' Forum: "A Note on Simplified Single Spool Turbojet Equilibrium Operation," by Charles A. MacGregor; "Plasticity Coefficients for the Plastic Buckling of Plates and Shells," by Alexander Krivetsky; "Comparison of Mixing Processes in Subsonic Jet Pumps," by Heinrich B. Helmbold; "A Simplified Treatment of Partial Fractions for Dynamic-Response Calculations," by George S. Campbell; "On the Possibility of Aeroelastic Reversal of Propeller Blades," by F. W. Niedenfuhr; "Heat Conducting Bodies in Boundary-Layer Flow," by Salomon Levy; "On the Calibration of Supersonic Wind Tunnels," by Jacques A. F. Hill; "The Behavior of the Boundary Layer in the Region of Transition from Laminar to Turbulent Flow," by Jerome Persh; "On Compressible Laminar Boundary-Layer Characteristics in a Pressure, Gradient with Heat Transfer," by Morris Morduchow ... 431–444

ST. LOUIS PUBLIC LIBRARY JUN 1955

Published by
INSTITUTE OF THE AERONAUTICAL SCIENCES, INC.

JOURNAL OF THE AERONAUTICAL SCIENCES

Editor: HUGH L. DRYDEN, *Director, NACA*
Managing Editor: RITA J. TURINO
Associate Editor: ROBERT R. DEXTER
Editorial Assistant: CAROLINE TAYLOR

Editorial Committee

Aerodynamics

Dean R. Chapman
Francis H. Clauser
Antonio Ferri
Alexander Flax
R. Paul Harrington
R. T. Jones
Carl Kaplan
H. W. Liepmann
C. C. Lin
H. F. Ludloff
Harold Luskin
John R. Markham
Clark B. Millikan
Shatswell Ober
W. Bailey Oswald
C. D. Perkins
Elliott G. Reid
H. J. E. Reid
Russell Robinson
George S. Schairer
G. B. Schubauer
W. R. Sears
John Stack
Theodore Theodorsen
Th. von Kármán

Air Transport

Charles Froesch
J. A. Herlihy
R. D. Kelly
Otto E. Kirchner
Jerome Lederer
John C. Leslie
R. Dixon Speas

Flight Testing

R. R. Gilruth
Melvin N. Gough
W. L. Howland
W. F. Milliken, Jr.
E. G. Stout

Structures and Materials

W. B. Bleakney
E. W. Conlon
John E. Duberg
E. C. Hartmann
N. J. Hoff
Samuel Levy
J. F. McBrearty
Alfred S. Niles
W. Ramberg
F. R. Shanley
C. R. Strang
S. Timoshenko
H. S. Tsien

Rotating Wing Aircraft

F. B. Gustafson
K. H. Hohenemser
Bartram Kelley
R. B. Maloy
Ralph McClarren
R. H. Miller
R. H. Prewitt
Igor I. Sikorsky

Vibration and Flutter

Lee Arnold
M. A. Biot
R. L. Bisplinghoff
William Bollay
Chieh-Chien Chang
J. P. Den Hartog
I. E. Garrick
Martin Goland
Theodore Theodorsen

Aircraft Design

W. E. Beall
E. H. Heinemann
Hall L. Hibbard
Richard Hutton
C. L. Johnson
A. A. Kartveli
E. G. Stout
F. K. Teichmann
Robert J. Woods

Electronics in Aviation

Welcome W. Bender
K. C. Black
M. V. Kiebert, Jr.
David S. Little
S. B. Rogers
Paul Rosenberg
L. M. Sherer

Physiologic Problems

H. G. Armstrong
Louis H. Bauer
Robert J. Benford
W. Randolph Lovelace, II
Ross A. MacFarland

Fuels and Oils

D. P. Barnard
J. H. Doolittle
Graham Edgar
R. T. Goodwin
A. M. Rothrock

Flight Propulsion

George W. Brady
Frank W. Caldwell
L. S. Hobbs
E. E. Johnson
Joseph Keenan
R. P. Kroon
Hans Reissner
Abe Silverstein
C. Fayette Taylor
E. S. Taylor
P. B. Taylor
E. S. Thompson
H. S. Tsien

Instruments

Allan G. Binnie
Charles H. Colvin
Frank R. Cook
C. S. Draper
S. G. Eskin
O. E. Esval
C. F. Savage
R. C. Sylvander

Meteorology

C. F. Brooks
D. M. Little
F. W. Reichelderfer
K. C. Spengler

SUBSCRIPTION RATES

JOURNAL OF THE AERONAUTICAL SCIENCES, Published Monthly.—United States and Possessions: 1 Year, $12.00; Single Copies, $1.50. Foreign Countries Including Canada (American Currency Rates): 1 Year, $13.00; Single Copies, $1.50.

AERONAUTICAL ENGINEERING REVIEW, Published Monthly.—United States and Possessions: 1 Year, $3.00; Single Copies, $0.50. Foreign Countries Including Canada (American Currency Rates): 1 Year, $3.50; Single Copies, $0.50.

Notices of change of address should be sent to the Institute at least 30 days prior to actual change of address.

Manuscripts for publication, proofs, and all correspondence should be addressed to the Editorial Office of the JOURNAL.

Correspondence regarding subscriptions may be addressed to Publication Office, 20th and Northampton Sts., Easton, Pa., or to the Editorial Office of the JOURNAL OF THE AERONAUTICAL SCIENCES, 2 East 64th Street, New York 21, N.Y.

Copyright, 1955, by the Institute of the Aeronautical Sciences, Inc.

Entered as Second Class Matter at the Post Office, Easton, Pa., May 1, 1937. Acceptance for mailing at a special rate of Postage provided for in the Act of August 24, 1912. Authorized April 29, 1937.

JOURNAL OF THE AERONAUTICAL SCIENCES

VOLUME 22 JUNE, 1955 NUMBER 6

Fatigue Life of Airplane Structures

The Eighteenth Wright Brothers Lecture

BO LUNDBERG*

The Aeronautical Research Institute (FFA), Sweden

SUMMARY

The dominating difficulty in the increasingly important problem of fatigue in airplane structures is the very great scatter that is typical for all its main aspects. Due to the fact that this scatter is, to a large extent, of an inherent nature and thus cannot be eliminated, it will never be possible to predict the fatigue life for an individual airplane with reasonable accuracy.

Consequently, our efforts should be concerned not with the prediction of fatigue life, but with the prevention of fatal fatigue failures down to an extremely low failure rate. A result of this general philosophy is the concept of a "limit" fatigue life defined as the life, in miles or hours flown, up to a "limit" probability level for fatigue failure. The combination of the limit life and its corresponding limit probability level should be such that the failure rate resulting from this combination has a satisfyingly low value, which should, if possible, be agreed upon internationally.

With these general thoughts as a background, the paper presents a survey of the various aspects of the fatigue of complete airplane structures, together with numerical applications. The survey is to some extent supported by test results obtained in Sweden. A general formula has been developed for the calculation of the cumulative damage. This formula has been used for the preparation of some charts for the calculation of the fatigue life as a function of the most important design and fatigue parameters. A new method by Weibull for the determination of *S-N*-curves at various probability levels is discussed and applied. The paper deals mainly with fixed-wing transport aircraft, although the general views and methods can be applied to all types of airplanes.

Finally, it is emphasized that, due to our highly insufficient knowledge of the very complex problem of fatigue in aeronautics, the survey presented has the nature more of a frame of the problem than of a complete picture.

FOREWORD

An invitation to present this lecture in commemoration of the Wright Brothers' historical achievement is indeed a profound honour under any circumstances. The honour is, I feel, further magnified because this appears to be the first occasion on which someone outside the United States or Great Britain has been invited to deliver this lecture. I interpret this distinction to imply an appreciation of the development of aviation in Sweden and in particular of the aeronautical research work carried out at the FFA.

I wish to express my sincere gratitude to the Council of the Institute of the Aeronautical Sciences for having been selected to present this lecture.

Presented before the Institute of the Aeronautical Sciences in the U.S. Chamber of Commerce Auditorium, Washington, D.C., December 17, 1954.

* Director.

CONTENTS

1. *Introduction.*
2. *Symbols.*
3. *Presentation of the Problem in General Terms.*
   - 3.1. Fatigue versus static strength; scatter disregarded.
   - 3.2. The nature of the fatigue problem for the case of no scatter.
   - 3.3. Scatter and probability.
   - 3.4. Fatigue versus static strength considering scatter.
   - 3.5. Failure rate.
   - 3.6. The probability factor.
   - 3.7. Summing up the problem.
     - Topic A. Designing for fatigue.
     - Topic B. Checking fatigue properties.
     - Topic C. Fatigue metering.
     - Topic D. Inspection and repair.
4. *Level of Safety.*
   - 4.1. General.
   - 4.2. The acceptable failure rate.
5. *The Gust Loads.*
   - 5.1. The gust load equation.
   - 5.2. The gust load spectrum.
6. *P-S-N-Information.*
   - 6.1. General.
   - 6.2. Some statistical concepts.
   - 6.3. The *S-N*-function.
   - 6.4. Different *P*-cases.
   - 6.5. Results of tests with large sample sizes.
   - 6.6. Pooling test results.
   - 6.7. Studies of the parameters in the *P-S-N*-function.
7. *Cumulative Damage and Life Calculations.*
8. *Designing for Fatigue; Topic A.*
   - 8.1. General.
   - 8.2. A general chart for fatigue life.
   - 8.3. Fatigue life diagrams based on tests.
   - 8.4. Design procedures for various types of design.

9. *Checking Fatigue Properties; Topic B.*
9.1. General.
9.2. A general chart.
9.3. An empirical diagram for two load levels.
9.4. An approximate formula and diagram for one load level.
10. *Fatigue Metering; Topic C.*
10.1. General.
10.2. Mechanical, strain actuated fatigue meters.
11. *Inspection and Repair; Topic D.*
12. *Fatigue Requirements.*
13. *Concluding Remarks.*
14. *Acknowledgments.*
Tables.
References.
Appendix A.
Discussion.
Reply.

1. Introduction

When the Wright Brothers created and flew the first power-driven and controllable aircraft in 1903, the problem of fatigue in the structure of this airplane was certainly not one of their main worries. Likewise, during the two or three decades subsequent to the first flight, the possibility of fatigue failures of vital parts in the airframe was hardly considered a problem, at least not if vibratory loads and cases of bad design causing undue stress concentrations were excepted.

In about the last two decades, however, there has been a growing trend of opinion that the accumulated effect of fatigue due to repeated, but mostly quite moderate, structural loads could sometimes cause failures of the primary structure before an airplane attains a reasonable or desired service life, even with good designs avoiding high stress concentrations. To begin with this reasoning was of a rather academic or speculative nature insofar as it was not, for a long time, supported by experiences of any tragic occurrences in the form of fatal fatigue failures. The opinion seems mainly to have been based on estimates of the accumulated damaging effect on the airframe of the manoeuvring and gust loads, which could be anticipated during a normal service life.[1] According to this line of thought compliance with the requirements for static strength was not a sufficient safeguard against serious fatigue trouble, and it was pointed out more or less expressively[2,3] that the general trends in aeronautical development, in particular towards higher speeds, greater structural efficiency and longer service life of the airplanes would make the fatigue danger more and more acute unless special actions were taken to prevent such failures.

It was quite natural, however, that as long as no serious fatigue troubles were actually encountered in service this trend in opinion was mostly considered to be unduly pessimistic. The more common belief was that fatigue in the primary structure was really nothing to bother about as long as one took care to avoid high stress concentrations in the design and to perform the routine inspections with a reasonable degree of thoroughness. This attitude was in a way understandable because there were so many seemingly more serious problems and worries that called for immediate solution and attention in order to improve safety, such as operational matters and climb performance.

Since the middle of the forties some transport airplanes in different parts of the world have suffered fatal fatigue failures of their primary structure. These occurrences, although they could to a considerable extent be attributed to unduly high stress concentrations, seem to have caused a marked change in the opinions regarding the importance of fatigue in fixed wing aircraft. Very extensive studies have been made of the fatigue problem in its entirety and its many important detail aspects have attracted intense research efforts. Investigations of the first mentioned general kind have usually dealt with the influence of all the main factors affecting the disposition of airplanes to develop fatigue failures and the trend in development of these factors. Such studies[4] have been of great value not only because they have shown that the trend in aeronautical development during the last one or two decades has increased very heavily the potential fatigue danger, but also because they have indicated that some of the trends in development will have to be changed from now on unless we are going to be put in a very difficult situation in the future. The intensified research efforts are mainly being devoted to obtaining statistical data on gust loads, to laboratory testing of various kinds of specimens and to verifying or modifying the theory for cumulative damage. Furthermore, the efforts of the physicists to discover the real cause of the fatigue phenomenon are followed with great expectations and in later years increased attention has been devoted to the statistical aspect of the fatigue properties of structural materials.

These studies and research work have resulted in a rather unanimous opinion that fatigue in fixed wing aircraft is a highly important matter that has to be considered for all types of transport airplanes. In the last few years there are also many indications that fatigue can be of great importance even for military aircraft, such as fighter planes, in spite of their short service life. It is also commonly agreed that our knowledge about all branches of the intricate problem of fatigue in airplane structures is in its infancy and we have consequently a very long way to go before we can tackle fatigue in aeronautics in a fully rational and reliable manner, be it in the design of airplanes or in their operation.

A natural consequence of our lack of knowledge in this field and the appreciation of its importance is a rather wide diversity of opinions about the most appropriate general policy that should be followed in the future as well as about the more immediate steps that should be taken to cope with the current fatigue difficulties. As is well known, one main line of thought favours the use of highly redundant structures—e.g., as with multispar wing construction. Pursuant thereto, the successive replacement of members which during inspection have been found to have suffered fatigue

cracks, is considered as a rather normal or at least acceptable method to assure a safe fatigue life. According to a contrary opinion a safe fatigue life might well be attained without resorting to structures of high redundancy, if only the fatigue sensitive joints are designed to comply with certain criteria. A further feature of this policy is that the service lives of the airplanes are scheduled according to life estimates at reasonable time intervals determined on the basis of laboratory tests, partly performed since the type has been put in service, and on gust measurements.[5] Some holders of this opinion seem to consider replacements of vital parts as a rather abnormal procedure, that as a rule should not be consciously counted on in the design of a new airplane.

It is felt that both these policies, if applied with great caution and with emphasis on thorough inspections at short intervals, can make the risk of catastrophic fatigue failures of airplanes now in service or under construction "extremely remote." This would mean compliance with the present rather vague ICAO and CAR requirements, at least if the concept of "extremely remote" is not too pretentiously interpreted. Another question is whether or not for the future any of these two main policies without further development can be considered a sufficient guide, either in respect to their suitability as bases for efficient designs together with economical operation, or as to their assurance of a sufficiently high level of safety seen in a world-wide sense. It implies no criticism to state that, at the present insufficient level of knowledge regarding aeronautical fatigue, both the policies, as they have been applied hitherto, are of the nature of temporary expedients making the best of a bad situation. Irrespective of the fact that extended research will considerably decrease the fatigue difficulties whatever policy is being followed, there now seems to be an urgent need to establish a universally accepted philosophy as a background for any special policy or method chosen to be used. Only by means of such a "common denominator" will it be possible in respect of fatigue to determine the most efficient ways to design new airplanes, as well as the most urgent needs for research, and to attain a universal agreement about requirements that can provide a uniform and sufficiently high level of safety.

This lecture is dealing in part with the question of such a general philosophy but its main purpose is to provide a technical survey of the aeronautical fatigue problem as it can be visualized to-day, in order to facilitate sound judgments of the future situation and needs. In order to achieve this goal the lecturer has attempted to treat the problem mostly from the designer's, and to some extent also from the operator's, viewpoint. Furthermore, the study has, in some respects, been made in a quantitative way, because it is felt that a discussion of the influence of all the various factors of importance for the fatigue problem in a merely qualitative way or merely with numerical examples, does not sufficiently clarify the problem.

With this outlook on the fatigue problem the paper has been deliberately limited in its scope; it thus is not claimed in any way to be a complete survey of the whole aeronautical fatigue problem. Such questions as the relationship between trends in aeronautical development and the increased disposition for fatigue in airplanes will not be dealt with here and many of the general viewpoints usually expressed in overall studies of the fatigue problem will be omitted in this presentation.

A further limitation will be, that although most of the views and methods presented in the paper are applicable also for military and private airplanes, the paper is mainly written with fixed-wing transport aircraft in mind and all the numerical applications are valid for this category of airplanes and in particular for their wing structures.

2. Symbols

2.1. *General Symbols*

a = slope of wing lift curve, per radian
B = parameter of S-N equation
b = parameter of S-N equation
c = parameter of S-N equation
c_P = probability factor $= c_{P_1} c_{P_2} = P_E/P_L$
D = fatigue damage
D' = damage intensity
E = endurance limit
F = failure rate
H = total number of gust cycles giving accelerations equal to or greater than a certain magnitude. One cycle is defined as one positive and one negative gust of the same magnitude
h = slope of gust curve, Eq. (24)
$I(u, p)$ = incomplete Gamma-function
i = number
j_I = safety factor based on load
j_T = safety factor based on time
K = alleviation factor
K_t = stress concentration factor
k_A = fatigue factor
k_{Af} = fictive fatigue factor, Eq. (26)
L = fatigue life in miles
L_a = absolute load amplitude
L_u = absolute ultimate strength
L_m = mean absolute load or 1g load
L_{max} = maximum absolute load
l_a = comparative load amplitude $= L_a/L_m$
M = average number of element failures in a structure
m = number of elements in a structure
N = number of load or stress cycles for failure
n = number of test specimens
n_D = ultimate design load factor in the most critical static load case
n_E = element load factor $= L_u/L_m$
Δn = load factor increment due to gust
n_i = number of applied load or stress cycles of a certain amplitude
P = probability of failure
q = number of components
R = risk of exposure $= dP/dT$
r = ratio R/F
S = stress or wing area, sq.ft.
s_a = relative load or stress amplitude
s_m = relative mean load or stress
s_{max} = relative maximum load or stress

T = operational time of an airplane in hours; fatigue life in hours
t_i = inspection interval in hours
U = equivalent vertical gust velocity, ft./sec.
V_e = equivalent air speed
v = relative equivalent air speed
W = weight of airplane
α = parameter of S-N equation
β = parameter of S-N equation
Γ = Gamma-function
μ = relative wing weight, (Figs. 10 and 11)
ν = relative gust sensitivity
ρ_0 = standard air density at sea level, slugs/cu.ft.
ω = relative weight of the airplane related to the design weight
ω_a = factor representing the effect on s_a of a weight reduction
ω_m = factor representing the effect on s_m of a weight reduction

2.2. *Statistical Symbols*

k_P = factor for calculating x_P with a certain confidence
n = number of test specimens
P = probability
s_x = standard deviation of a variable x computed from a limited sample size
U = normal deviate
v_x = coefficient of variation of a variable x
x = statistical variable
$\bar{x}$ = mean value of x computed from a limited sample size
x_{50} = median value of x
x_P = value of x corresponding to a probability P
γ = confidence coefficient
σ = true value of standard deviation of a variable (at infinitely large sample size)

2.3. *Subscripts*

A = area
a = amplitude
b = basic
c = centroid
cr = critical
D = design
E = endurance limit or element
e = equivalent
f = fictive
i = inspection
L = limit
l = comparative load
m = mean
P = probability
r = reduced
u = lower limit or ultimate

3. Presentation of the Problem in General Terms

3.1. *Fatigue Versus Static Strength; Scatter Disregarded*

Let us first disregard the abundant scatter and other uncertainties that to such a high degree characterize all the main aspects of fatigue. Regarding the fatigue qualities of a structural element this would mean that a number of nominally identical elements would fail after exactly the same number of load cycles above the endurance limit. Considering a number of airplane wings built to the same specification and assuming that they are subjected to varying loads following identical load spectra, absence of scatter would imply that

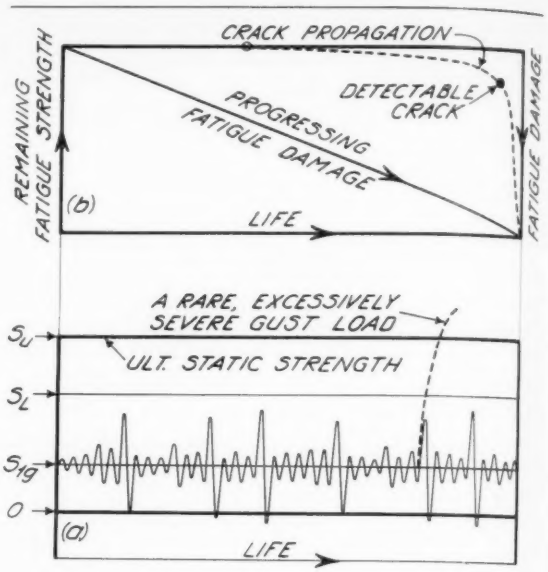


Fig. 1. Schematic comparison between fatigue and static strength; scatter in fatigue disregarded.

the planes would fail after precisely the same life, for instance expressed in miles flown. Under the assumption of no scatter, a fundamental difference between the static and the fatigue sides of structural strength is evident. As is well known, static failure is mainly a matter of the chance of encountering a high load, for instance a severe gust load, in excess of the ultimate static strength of the structure. It is in other words at least basically a question of probability as illustrated in Fig. 1a. Fatigue failure on the other hand, under the said assumptions, is merely a matter of the application of a sufficient number of load cycles or of the airplane flying a certain total distance.

The implication of the fatigue phenomenon can be visualized by defining the "remaining fatigue strength" as a quantity proportional to the remaining fatigue life, both being 100 per cent before the structure has been subjected to any loadings above the endurance limit, as illustrated in Fig. 1b. The structure is consequently *bound to fail* when its remaining fatigue strength reaches zero—i.e., when its fatigue life reaches its ultimate value. This occurrence obviously has the character of an inevitable destiny and is not a matter of probability under the assumed conditions of zero scatter in fatigue strength and applied load spectrum.

This difference in the nature between fatigue and static strength qualities is of course self-evident for anyone familiar with fatigue problems. Nevertheless, it is important to emphasize this difference for those who have not specialized in fatigue matters, because there still seems to be a fairly common feeling that a structure, even if it has been subjected to repeated loads above the endurance limit, remains undamaged as long as no detectable or non-detectable, fatigue cracks have developed. It appears to be an unsettled question

between the physicists, whether or not a submicroscopic crack develops at the very first application of a load. However, even if such a submicroscopic crack does not start until the specimen has been subjected to a certain number of load reversals—i.e., even if the crack propagation follows the general pattern illustrated by the dashed curve in Fig. 1b, the damage should be considered or defined as proportional to time (at a given pattern of load cycles), following a straight line relationship, as shown in this figure. This concept of proportionality between damage and life might only be a matter of definition, but the definition seems sound and correct provided that the pre-crack damage is not eliminated by the material being able to regain its initial properties purely by resting, which to the knowledge of the lecturer has not been proved to be the case, at least, not to any appreciable extent. Fatigue damage and crack-propagation are thus two different matters which for sake of clarity should be kept apart.

Another question is how the actual fatigue failure should be defined. From a purely physical point of view failure means, of course, that the element or part in question can no longer sustain the applied loads. This simple definition is quite satisfactory for instance for parts of mechanisms which merely are subjected to fluctuating loads of a given pattern, without having to be able to withstand occasional "static" loads appreciably higher than the fluctuating ones. Under such loading conditions a complete physical failure will be the ultimate outcome of the repetitions of the loads, provided they are repeated a sufficient number of times. For an airplane wing the situation is complicated by the fact that the structure, according to the requirements for static strength, must be able to withstand at any time a single static load very much higher than the majority of the varying loads. A strict compliance with the static strength requirements would therefore lead to the concept of an "allowable fatigue crack," which would mean that a load-carrying member should be considered as having failed by fatigue as soon as the fatigue crack has reduced the static strength of the complete structure below that specified by the static strength requirements. This in principle would mean that fatigue cracks are allowed to occur only when, and to the extent that, the structural component in question is statically overdimensioned. As over-dimensioning of fatigue-sensitive parts, such as joints, seems to be necessary in many cases, a rule of this kind would permit a certain propagation of fatigue cracks even in vital parts, provided that (1) the cracks during, or still more strictly at the end of, a flight have not grown beyond the extent at which the static strength requirements are just complied with, and (2) the cracks are detected and the damaged parts repaired before the next flight.

However, such a rule might seem too severe, considering the extremely small probability of the airplane encountering a gust or manoeuvring load exceeding the required ultimate strength of the airplane during the very same flight in which the cracks grew to the critical depths. It therefore could be justified to allow fatigue cracks to cause a certain reduction of the static strength below the minimum stipulated, but such an allowance must be made with extreme care and with due consideration of the rapidity of crack propagation and the detectability of the cracks. It might be borne in mind that in one of the most disastrous structural failures that have occurred, the wing spar finally broke in a thunderstorm after its static strength had been considerably reduced due to a highly developed fatigue crack.

A general comparison of static and fatigue strength would be rather incomplete without touching upon the interesting question of whether failures of the two kinds can be grasped by one single theory. A number of scientists have outlined theories in this direction and some of them seem to feel that such a comprehensive theory would be of importance also for practical purposes. It has even been hinted that a single theory would make it possible to utilize the vast experience that has been obtained in the past regarding static strength and safety as a guide that would help us to tackle the future fatigue problem.

This "one theory" approach might be desirable as far as basic research is concerned, dealing with the problem of the causes of fracture due to any number of loads. It might nevertheless be pointed out that an essential difference will always be present, because static failure takes place after the yield point of the material has been passed, whereas this is not usually a typical occurrence at fatigue failure. Regardless of this it is strongly felt that for the designer the "one theory" line of thinking, for another reason, is hardly of any help at all, and that it might even be confusing. The reason is that the *number* of load cycles, which for any given load spectrum is a measure of operational *time*, is the determining factor for fatigue failure, whereas the *magnitude of one single high load* is the determining factor for static failure. This fact will always distinguish the two types of failure and the corresponding two fields of design as two widely different topics. This statement might be regarded as a matter of philosophy, but it is felt that a sound philosophy should be the very basis for attacking engineering problems. The significance of the fundamental difference between the static and the fatigue sides of structural analysis comes out clearly when dealing with the important matter of safety.

3.2. *The Nature of the Fatigue Problem for the Case of No Scatter*

Let us now briefly present the comparatively simple problem with which the designer would be confronted, if we maintain the assumption that the scatter is zero or so small, that it could be disregarded. The task of the designer would then be to design the airplane in such a way that it attains a certain desired or specified fatigue life when subjected to a prescribed or assumed spectrum of varying external loads. It is usual to illustrate this situation by drawing two curves

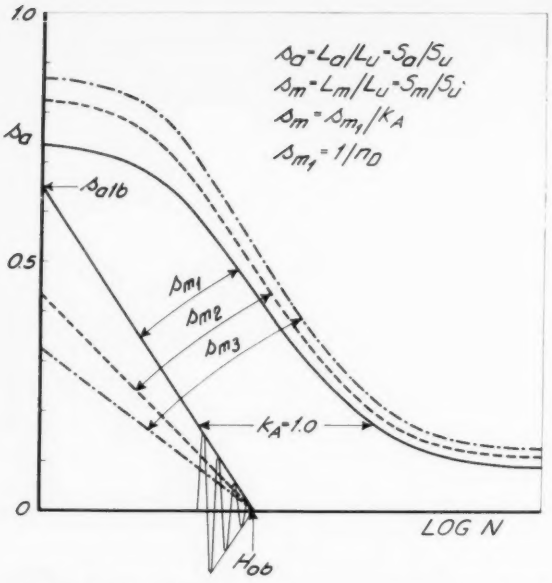


Fig. 2. Illustration of corresponding load spectrum and endurance curves at different relative mean loads.

(Fig. 2) with the varying loads plotted as functions of the number of load cycles, one representing the varying applied loads or load spectrum under a certain period of operation, and the other being the endurance or S-N curve of the element or component, the fatigue life of which has to be determined. For both curves the ordinates can be expressed in a number of ways using either absolute or relative quantities. For general treatments of fatigue of wing structures the most practical way of expressing the varying loads is in terms of the relative load (or stress) amplitude obtained by dividing the absolute load (or stress) amplitude by the absolute ultimate strength, thus

$$s_a = L_a/L_u = S_a/S_u$$

The course of both the curves is dependent on the mean load, which should, to be consistent, be expressed as the relative mean load

$$s_m = L_m/L_u = S_m/S_u$$

In order to calculate the cumulative damage effect of the varying external loads, it is of course necessary to consider related curves or functions for both the load spectrum and the endurance curve—i.e., the pair of curves or functions shall be valid for the same relative mean load. This is illustrated in Fig. 2, in which three pairs of curves are drawn for three values of the relative mean load. One of these mean loads,

$$s_{m_1} = 1/n_D$$

corresponds to the minimum ultimate design load factor, n_D, just necessary for compliance with the static load requirements.

For obtaining the desired or required fatigue qualities, it is usually necessary to increase the ultimate static strength above the said minimum by increasing the structural area. This increase can be expressed by a multiplying factor, k_A, which in this paper will be called the "fatigue factor." An increase in area corresponding to k_A will of course reduce both the applied relative load amplitude and the applied relative mean load and thus increase the fatigue life. The relative mean load will be reduced from s_{m_1} to

$$s_m = s_{m_1}/k_A$$

An increase of the area corresponding to the factor k_A evidently implies an additional static strength on top of the minimum static requirements which are satisfied by the static factor of safety (normally 1.5) together with the limit load. This means that k_A, as far as static strength is concerned, is a measure of "the static margin of safety" which is equal to $k_A - 1.0$. It is important to emphasize, however, that, when dealing with fatigue, the factor k_A should not be regarded as a factor of safety and still less as an additional margin of safety. Under the assumption of no scatter, it is evident that the increase in area corresponding to k_A is an absolute necessity in order to provide a certain desired fatigue life. This desired fatigue life should be regarded as a "limit" condition corresponding to the required limit load on the static side. Ascertaining a specified limit fatigue life should obviously be considered as an equally important design criterion as attaining a certain static strength. For the hypothetical case of zero scatter no factor or margin of safety is necessary, either on top of the required or limit fatigue life or on the structural area corresponding to the fatigue factor k_A found necessary to provide the said fatigue life. In case of scatter in one or several of the aspects of the total problem of aeronautical fatigue the introduction of one or a number of factors of safety as regards the fatigue qualities is one possible way of ascertaining a required level of safety and this matter will be discussed in a following chapter.

Turning back to Fig. 2 it might be pointed out that decreasing the relative mean load by the factor k_A not only gives a decrease of the applied relative loads in direct proportion to k_A, but also improves the endurance curve, although normally to a rather moderate extent, as illustrated by the figure.

Under the assumption of no scatter, the designer's problem would be limited to the following main items:

1. To find or assume a relevant load spectrum and to express that in terms of s_m or k_A.
2. To determine a s_a-s_m-N family of curves for specimens closely representing the type of structural element contemplated for the design.
3. To apply an accepted cumulative damage theory on pairs of load spectrum and s_a-N curves, having the same relative mean load, with the object of finding that value of k_A or the mean load which gives the structural element a certain desired or required fatigue life.

Another recognized method of dealing with the problem is to carry out a series of "varying amplitude tests," which in principle means that a "unit" load

spectrum for each test is repeated until failure of the specimen occurs. The tests have to be carried out for a number of different values of s_m or k_A. A lowering of the mean stress level will give prolonged fatigue life, and thus that value of the fatigue factor, k_A, can be determined, which gives the desired fatigue life.

For pressure cabin loads the problem is inherently simpler because the load spectrum in principle is of the "constant amplitude" type—i.e., the type used in normal S-N tests according to their definition. As the lower limit for pressure cabin loads is zero, or close to zero, the S-N tests on which the design is intended to be based should also be carried out with the lower limit zero. Consequently the mean load, half of the maximum load, loses its meaning as a parameter in this type of endurance test and the maximum load, or preferably the relative maximum load or stress,

$$s_{max} = L_{max}/L_u = S_{max}/S_u$$

is the natural ordinate in plotting the curves.

As illustrated by Fig. 3, the s_{max}-N curve itself represents the fatigue life curve from which the fatigue factor can be determined in order to give a required fatigue life expressed in load cycles or flights. The fatigue factor is also in this case defined so that it is at the same time a multiplying static factor, the minimum value of which, according to the current requirements, shall be 1.33. This minimum value, however, is not necessarily sufficient to provide a certain desired fatigue life in numbers of flights.

3.3. *Scatter and Probability*

It has been known for a long time that all fatigue tests display a very large scatter in the results, even if extreme care is taken to manufacture a number of test specimens identical within very narrow tolerances and the testing machine is rigorously checked as to

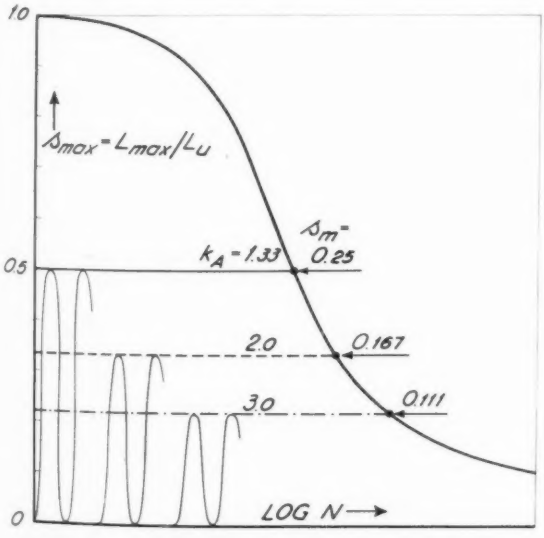


Fig. 3. Illustration of cabin pressure loads at different stress levels.

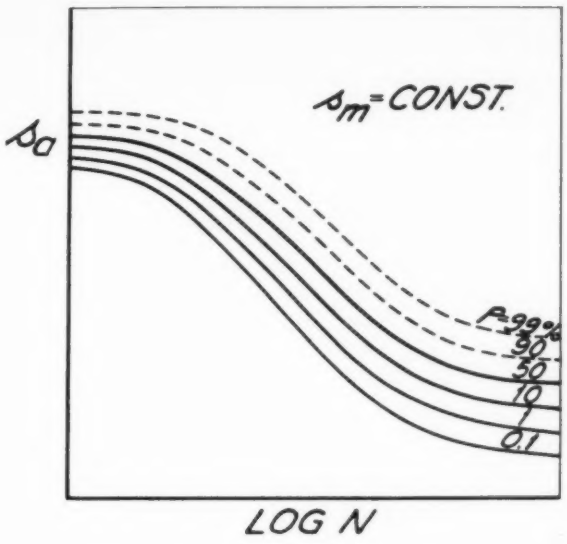


Fig. 4. P-s_a-N-diagram.

the magnitude of the applied loads. Much research has been and is being done by physicists and metallurgists in order to determine the physical cause for fatigue or to find theories or hypotheses which fit the physical phenomena of fatigue, but so far, only little success has been obtained. Should these efforts eventually yield a satisfactory explanation or interpretation of fatigue, this might lead to the development of materials with improved fatigue properties compared with those which characterize some of the light alloys now commonly used in aeronautics, and this is, of course, a highly desirable target to aim at. That successful basic research of this kind, however, would lead to an appreciable reduction of the scatter seems highly doubtful in view of what we already know about the submicroscopic nature of the behaviour of metals subjected to repeated loads. A quite considerable scatter in the fatigue properties must therefore be regarded as an inherent characteristic of all materials, as well as of structural elements, built-up parts and complete components, and it is believed that this state of affairs must be accepted as an inevitable fact also for the future. This implies that the only rational way of dealing with the fatigue problem is by means of statistical methods, which in turn means that the concept of probability must be introduced into the picture.

As regards the S-N side of the problem, the situation is illustrated by Fig. 4, which shows a typical scatter band in s_a-N-plot with P-curves for various probability levels. P here signifies the probability of failure and is defined as

$$P = i/n \qquad (1)$$

where i is the number of specimens that fail and n is the sample size. It might be pointed out here that the probability concept is always tied up with average conditions and furthermore that the sample size must be about as large as indicated by the P-level under

consideration. Thus the determination, with good accuracy, of a $P = 1$ per cent level presupposes a sample size of about 100 and to determine $P = 0.1$ per cent about 1,000 specimens would need to be tested.

So far the word scatter has been used as a measure of the varying results that are obtained in S-N-tests carried out in a laboratory with nominally identical specimens and loading conditions. The structural elements in an airplane, however, are normally manufactured to much wider tolerances than test specimens. They are subjected to external wear and tear during service and the loads applied to them might change during the operation of the airplane due to redistribution of the load pattern, for instance as a result of some very high loads.

These and other similar factors tend to increase the scatter band for an airplane as compared with that for a laboratory element, and this is why great care must be taken in transferring scatter in S-N-results from laboratory tests to scatter in the fatigue properties of the complete airplane.

Also the two other main aspects of aeronautical fatigue, the load spectrum, in particular for gusts, and the cumulative damage, are afflicted with quite large uncertainties or matters of ignorance and even scatter in a more limited statistical sense. The gust loads depend appreciably on altitude, the geographical location of the routes and on the operational technique, etc. The resulting possible outer limits for the gust load spectrum differ by a factor of the order of 20 or more (see Fig. 12, Chapter 5). It should be emphasized, however, that all this spread is not "scatter" in quite the normal meaning, because we can very well neglect the possibility of an individual airplane being used for operations widely different from the type of operation the airplane is designed for—e.g., a long range high altitude transport plane will hardly be used for very short-distance flying. The actual scatter band of the load spectrum or load history for planes of a certain type will therefore always be narrower than the spread-band corresponding to the gust distribution in the whole atmosphere, but nevertheless, the scatter band in most cases will be distressingly wide.

The only method of radically decreasing the uncertainties in load history is by recording for each individual airplane, preferably continuously, the load history or the damaging effect of the loads encountered. A general adoption of such a recording technique, by installing "fatigue meters" of one kind or another, would bring about quite a substantial reduction of the total uncertainty in applied loads. However, even with the best conceivable types of fatigue meters, there will always remain an appreciable scatter or uncertainty about the applied loads, as no type of fatigue meters can represent exactly the loading conditions for all the fatigue-sensitive parts of an airplane.

The uncertainty with respect to the cumulative damage at varying load patterns is at present, to a large extent, a matter of ignorance, as we still know very little about the cumulative damage effect and as the validity of the hitherto commonly used theory ($\Sigma n/N = 1$) is questionable. Regarding the future situation in this respect, it might be pointed out that even if further research leads to a more accurate theory, considerable scatter will still remain, as the cumulative effect has been proved to be dependent on the sequence in which high and low loads follow each other, for instance whether a very high load is encountered in the beginning or later on in the service life of an airplane. This source of scatter obviously cannot be eliminated even if the fatigue life for a complete airplane were studied by varying amplitude fatigue tests.

This review of scatter and other variabilities in aeronautical fatigue is by no means complete. A number of other factors might also contribute more or less markedly to the total amount of uncertainty. One of these factors is the increase in the probability of failure in the case where the airplane structure is built up by a number of fatigue-sensitive parts. This is an intricate and important problem which will be dealt with further on. Another, although related, cause of uncertainty is the fact that a structural member of an airplane might be subjected to a different loading pattern as compared with that applied in laboratory tests. The fatigue tests in an ordinary fatigue machine are normally arranged so as to give a constant stress or load level even after a considerable fatigue crack has developed. In an airplane structure, however, the development of a crack in a part which is a member of a statically undeterminate structure, might cause a decrease of the load on that part simultaneously with its area, and hence its stiffness, being reduced by the propagating crack. This decrease of the load on the member might make the loading conditions correspond more closely to loading with constant strain amplitude instead of constant stress. This and similar circumstances call for extensive research, but it will probably always be difficult to take them accurately into account.

From the above discussion it follows that scatter and other uncertainties are inevitable, and largely inherent, characteristics in aeronautical fatigue. The total scatter will certainly be decreased by extended research and improved techniques, but it is believed that it will never be possible to reduce all the scatter in fatigue life below a value of several hundred per cent. The conclusion to draw from this state of affairs cannot, the author thinks, be expressed more clearly than has been done by Rhode at the Fourth Anglo-American Conference, 1953:[4] "In short, present inability to calculate the fatigue life is such that any number coming from such calculations and purporting to represent the fatigue life on an absolute basis is meaningless." It is believed that this statement will always be valid but, on the other hand, this does not imply any reason for pessimism providing we change the target to be aimed at from attempts to predict the fatigue life of an individual airplane and instead direct our efforts to preventing the occurrence of fatal fatigue failures.

3.4. *Fatigue Versus Static Strength Considering Scatter*

It was pointed out in Section 3.1 that in the case where scatter could be disregarded the occurrence of a static failure of an airplane, due for instance to a severe gust, is by nature a question of probability, whereas the fatigue failure would not be a matter of probability as the structure would be bound to fail after a certain number of load cycles. However, the fact that in reality the total scatter in aeronautical fatigue is of an unusually high magnitude, undoubtedly makes fatigue in the airplane a very pronounced probability question. This certainly modifies, but it does not in principle change, the fundamental difference between fatigue and static loading and failure, which can now be summarized as follows:

(a) Even considering scatter the significant quantity in fatigue, at a given loading pattern, is still the number of load reversals or operational *time* (whereas applied *load* is the important concept for static strength and failure).

(b) The nature of the probability aspect for fatigue loading is widely different from that for static loading. For any certain type of operation the probability of the airplane encountering one severe load, for instance a gust load, in excess of the static strength is in principle, independent of elapsed time, as such an occurrence might happen on any occasion during the operational life of the airplane. The probability of a fatigue failure, however, changes quite markedly during the service life of each airplane. This is illustrated by Fig. 5a which shows a typical probability or "distribution" curve. The probability of failure due to fatigue for any good airplane design is zero during the first flights, zero or negligible for a considerable part of the life time and subsequently it reaches a disturbing level of say 1 per cent, whereupon it would further increase to and above 50 per cent if not withdrawn from service (all under the assumption that no repairs are made). This is the characteristic feature of the probability aspect in fatigue, which differentiates it from most other probability questions in aeronautics.

As thus the probability aspect, although in different ways, comes into the picture for both the static and fatigue sides of airplane design and operation, it might be of some interest to compare the relative importance of the probability feature for the two cases. Considering only transport airplanes, it should first be observed that the magnitude of the "ultimate" velocity of a gust, which would break an aircraft wing, for one and the same "limit" gust velocity, varies both with the gust sensitivity and with the direction of the gust. If the gust sensitivity is expressed in terms of the "limit" acceleration increment caused by a 50 ft./sec. gust, and is such that this gust velocity causes an acceleration increment of 1.5, the static gust case coincides virtually with the manoeuvring case for upward loads, and then the ultimate positive gust velocity would be $50 \times (3.75 - 1)/1.5 = 92$ ft./sec. At the same sensitivity for downward gusts and provided that the required wing strength for downward manoeuvring loads is just complied with (ultimate negative acceleration = 1.5), the ultimate negative gust velocity would be $50 \times (1.5 + 1)/1.5 = 83$ ft./sec. Assuming then a higher gust sensitivity, for instance corresponding to a limit acceleration increment of 2.0, it is found that the ultimate velocities for upward and downward gusts are $50 \times (4.5 - 1)/2 = 88$ and $50 \times (1.5 + 1)/2 = 63$ ft./sec., respectively.

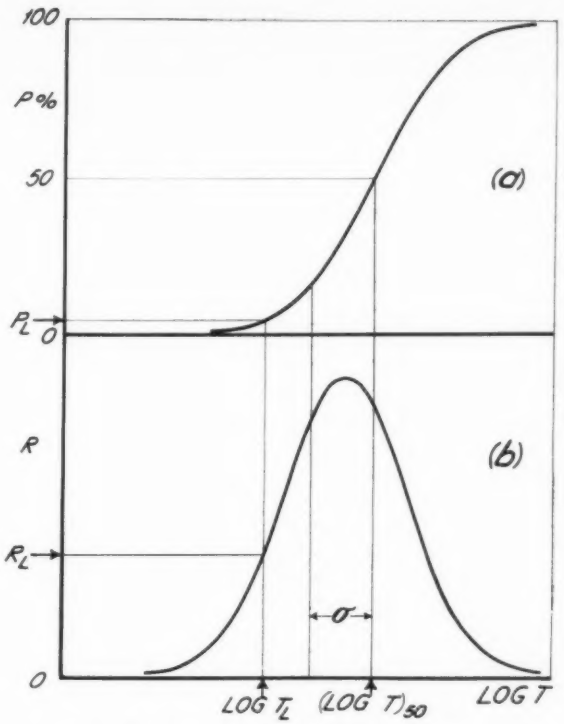


Fig. 5. Log normal distribution curve (a) and frequency curve (b).

As thus the magnitude of the ultimate gust velocity varies so considerably and as the frequency of gusts decreases very markedly with increased gust velocity, it follows that the probability of encountering a gust which would break the wing is highly dependent on the gust direction and on the gust sensitivity of the airplane. Because the ultimate gust velocity is smaller for downward gusts than for upward gusts, the former are normally the critical ones. If the discussion is limited to considerations of the overall safety, it is felt that a negative ultimate gust velocity of about 80 ft./sec. would be a representative basic value for estimating the overall probability of static failure of current types of transport airplanes, assuming that the increased probability of failure for the more gust sensitive planes is approximately balanced by the decreased risk for the less gust sensitive ones.

Even with such a fixed "critical level" of gust velocity, around 80 ft./sec., the probability of failure is difficult to determine due to the facts that the available statistical data regarding gusts of this magnitude are quite insufficient and that the available records of less

severe gusts disclose a very large spread depending on altitude, geographical location and source of information, etc. Extrapolations by eye of the gust spectrum curves shown in Fig. 12 indicate that one gust of around 80 ft./sec. on the average might be expected in a total distance flown of anything between 10^7 and 10^9 miles. Such simple extrapolations are naturally very inaccurate; for more reliable estimates statistical methods must be used, for instance by applying the theory of extreme values.[6] This method seems to indicate that the chance of encountering a 80 ft./sec. gust would be of the order of once in 10^7 to 4×10^7 miles—i.e., around the lower limit just mentioned.*

The accident statistics, however, do not seem to verify this comparatively high probability of static wing failure for transport airplanes. From 1946 to 1953 the scheduled domestic air services in the U. S. have accumulated around 3×10^9 plane miles with only one airframe failure that can be attributed to severe gust loads. Further, it might be pointed out that the official report of this accident[7] indicates that this failure could most likely have been avoided if a certain issued Severe Weather Bulletin had been received by the pilot.

This latter circumstance leads to the question of whether or not the attainment of exceedingly high static gust loads can be considered as purely a matter of probability in a statistical sense. It must be kept in mind that the airplane is not a robot but is operated by intelligent minds. It therefore appears that an exterpolation by statistical methods of recorded gust loads provides a too pessimistic picture for the future, in particular if one considers the ever-increasing meteorological knowledge resulting in improved means of avoiding regions of very turbulent air, such as thunderstorms. The author's feeling is that, with the present static requirements, the probability of static failure due to gusts is almost negligible, at any rate in the future, if one presupposes that improved means will be found to avoid the very rare, but severe, "clear air gusts" due to jet streams.

Static failure, however, can also be caused by a high manoeuvring load, for instance due to a hard pull-out in order to avoid collision with any object—e.g., due to bad visibility conditions. Structural failures of this type are also extremely rare, but it might be stated that not even these can be considered to be merely a matter of probability in a statistical sense. Anyhow the pilot should have a reasonable chance to prevent a catastrophe by making a hard pull-out without breaking the airplane. In view of these remarks a word of caution might be said in connection with the recent proposals made[8, 9] to consider a reduction of the present static factor of safety of 1.5.

* It is realized that this discussion of the probability of static failure is rather schematic as the scatter in static strength is assumed to be zero. A complete statistical treatment should of course also include this scatter[63, 64] but for the purpose of providing an overall picture the above discussion might be sufficient.

If we now sum up the comparison between the aspects of probability regarding static and fatigue failures, it can be stated that the probability of static failure of transport airplane wings is almost negligible and that the question of static strength should not merely be treated as a probability problem. The matter of fatigue failure, on the other hand, owing to the very large scatter, is such a pronounced probability question that the only rational way of tackling the problem is by means of statistical methods. It might furthermore be added that the accidents due to fatigue during the last decade indicate that at present the probability of fatigue failure, taking all existing types of transport airplanes into account, is considerably higher than that of static failure. How this will be in the future will, of course, entirely depend on what we are going to do about the fatigue situation.

3.5. *Failure Rate*

The tackling of the aeronautical fatigue problem by statistical methods necessitates some kind of governing philosophy as to a proper definition of the level of safety to be aimed at. The first thing to be agreed upon is whether the safety concept should be based on incidents or accidents. To use the occurrence of an incident does not seem very practical, in the first place, because it is a rather vague concept. It could be defined in a number of ways, for instance, as the occurrence during flight of a "detectable" crack, but then it remains to define what is meant with a detectable crack which, of course, must be dependent on the available, and applied, methods of inspection. Another possible definition of a fatigue incident would be the occurrence of cracks or failures of structural components so bad, that the static strength is reduced by a certain degree. Obviously any definition of an incident due to fatigue would involve a considerable degree of arbitrariness, and moreover, the risk of an incident leading to an accident of a catastrophic nature would differ greatly for different types of design, different operational conditions and different inspection procedures.

The occurrence of catastrophic failures seems to be a much more unequivocal concept and is therefore suggested to be used as the basis for treatment of the question of safety against fatigue.

The level of safety can, nevertheless, be defined in a number of ways, some of which will be briefly discussed.

One method would be to consider only the probability of failure, which is a function of the operational time

$$P = P(T)$$

The growth of P with time—i.e., the distribution function, is illustrated by Fig. 5a, assuming a log-normal distribution. A requirement that the probability of failure of a catastrophic nature should never exceed a certain maximum value, P_{cr}, the "critical" probability, would mean that P_{cr} would be that fraction of a number of airplanes, for instance a fleet of a certain type, that on the average would be allowed to suffer

fatal fatigue failures. Such a basis for the safety concept could possibly be of some value for limited applications—e.g., if one considers a fleet of military airplanes of a certain type or class, say all the fighter planes in a country.

The disadvantage of only considering the probability of failure is twofold, namely, that it does not give credit to a long fatigue life and that it does not indicate the level of safety in a more general way, for instance applicable for civil aviation as a whole. It might also be pointed out that a definition of this kind might possibly lead to applying different values of the critical probability for different sizes of the fleet of airplanes under consideration. As an example, such a high P_{cr}-value as 1 per cent might superficially seem satisfactory if the fleet only consisted of 10 planes, whereas this value evidently is much too high if 1,000 planes were considered.

Another way of considering safety would be to impose some demands on the maximum "risk of exposure" that would be permitted on flying with any airplane of a certain category—e.g., civil transport airplanes. The risk of exposure per hour of flight, R, is equal to the derivative of the distribution function

$$R = dP/dT \tag{2}$$

or identical to the frequency function illustrated in Fig. 5b.* Its highest permissible or "critical" value would correspond to a limit value of T, thus

$$R_{cr} = (dP/dT)_{T=T_L} \tag{3}$$

However, a sole application of the safety concept merely on a maximum or critical value of the risk of exposure does not seem to be satisfactory either, for broadly speaking the same reasons as those mentioned concerning the probability itself—namely, that R_{cr} does not yield the influence of the fatigue life, nor does it provide a picture of the overall level of safety.

A third and rather straightforward method would be to use the concept of failure rate, defined as

$$F = P/T \tag{4}$$

as a measure of the level of safety. The maximum permitted or critical failure rate would be

$$F_{cr} = P_L/T_L \tag{5}$$

In this equation T_L is the "limit" fatigue life corresponding to the concept of limit load, the word limit implying a permissible level or occurrence. On the static side an ultimate or design condition is defined by multiplying the limit load with a factor of safety. The application of the same method as regards fatigue life would lead to the definition of a design fatigue life, T_D, and a factor of safety on life, thus

$$T_D = j_T T_L \tag{6}$$

This factor of safety, j_T, would then cover, partly or wholly, the scatter and other uncertainties between the "limit" conditions (L) and the "design" conditions (D). The latter conditions can be defined in different ways; they can for instance be referred to the mean value of the total scatter band. This investigation will mainly deal with the principles for treating all the uncertainties by statistical methods, which means that no special factor of safety would be necessary. In case some or all uncertainties are left out from fatigue analyses one or a number of factors of safety have to be applied, a matter which will be discussed in Chapter 12.

The critical failure rate obviously is the mean value of the risk of exposure

$$F_{cr} = \frac{1}{T_L}\int_0^{T_L} \frac{dP}{dT}\,dT \tag{7}$$

up to the limit, or any other fixed, life. The concept of failure rate may also be applied as a measure of the average safety, both for all the airplanes of a certain type regardless of the utilization rate of the individual planes and for all airplanes of a certain class or category, for instance all civil transport airplanes, even if the different types of this class are designed to different failure rates. Thus

$$F = \Sigma P_m/\Sigma T_m \tag{8}$$

where P_m and T_m are the actual probability of failure and the corresponding life in hours, respectively, for one individual airplane. When n airplanes of the same type are utilized up to their limit probability and limit life, Eq. (8), will obviously be transformed into Eq. (5) as

$$\Sigma P_m/\Sigma T_m = nP_L/nT_L = P_L/T_L \tag{9}$$

The generally defined failure rate of Eq. (8) is obviously consistent with the terms in which accident statistics are presented as it gives the average number of failures per hour flown. It might be added that if one and the same critical failure rate is applied for all types of airplanes belonging to a certain category, for instance transport airplanes, the total resulting failure rate will obviously be equal to the failure rate for each of the types.

For the reasons stated, it is proposed that the failure rate should be considered as the basis for dealing with fatigue problems in aeronautics. This can, of course, be done without any general agreement as to an appropriate quantitative value of the critical failure rate. In the long run, however, it does seem highly desirable that at least all civil transport airplanes should be built to comply with one single, numerically determined level of safety, a matter which will be discussed in Chapter 4.

Although the failure rate seems to be the most appropriate basis for fatigue studies in aeronautics, it might possibly be desirable to employ also certain critical values of the risk of exposure or of the probability itself as further limitations. Dealing first with the risk of exposure, it is of interest to establish the

* It might be observed that in Fig. 5 the maximum of the frequency curve does not correspond to $P = 50$ per cent of the distribution curve because the abscissa of the diagrams is log T and R is the derivative with respect to T.

relationship between corresponding values of R and F if the life T of an airplane is represented by a log-normal distribution (Fig. 5), then

$$P = \Phi(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\xi} e^{-\xi^2/2}\, d\xi \qquad (10)$$

where

$$\xi = [\log T - (\log T)_{50}] / \sigma \qquad (11)$$

where σ is the standard deviation (see Section 6.2 and Fig. 5).

Eq. (2) gives

$$R = \frac{\log e}{\sigma T \sqrt{2\pi}} e^{-\xi^2/2} \qquad (12)$$

The ratio between R and F is thus

$$r = \frac{R}{F} = \frac{\log e}{\sigma P \sqrt{2\pi}} e^{-\xi^2/2} \qquad (13)$$

If P is given a fixed value P, the corresponding ξ for the normal distribution can be obtained from statistical handbooks.[10,11] The ratio r is then a function of the standard deviation, σ, exclusively, which is shown in Fig. 6 for P = 1, 0.1, and 0.01 per cent. From this figure it can be seen that the maximum risk of exposure per hour is of the order of 10 times the average failure rate. In view of the order of magnitude of the critical failure rate discussed in Chapter 4, the risk of exposure does not seem to be so high that it would be necessary to introduce an overriding limitation regarding risk of exposure on top of the critical failure rate.

To apply a certain maximum probability of failure as an additional limitation might, seen superficially, be of some value. A little afterthought, however, immediately shows that such an extra limitation would be meaningless, as P and F are interrelated according to the definition, Eq. (4).

In this connection it might be of some interest to clarify the practical meaning of probability and failure rate with an example. Say, for instance, that a fleet of 100 planes of a certain type are designed to have a limit fatigue life of 10,000 hours, implying a total flying time of 10^6 hours. If now a maximum failure rate of 10^{-8} were required, then this would mean that the type would have to be designed to a probability level not higher than $P = (TF = 10^4 \times 10^{-8} =) 10^{-4}$. This implies that on the average one of 10,000 airplanes would be expected to suffer a fatal fatigue failure, a ratio which might seem unnecessarily low in view of the fact that the fleet only comprises 100 planes. Nevertheless, this probability figure would be necessary to comply with if this particular fleet of airplanes were not to impair the overall level of safety represented by the demand in this example that the overall failure rate should not exceed 10^{-8}. Only if the fleet were increased to 10,000 planes instead of 100, would one plane of the fleet be rather likely to fail by fatigue, but then the total hours of flight would amount to exactly 10^8 hours, the inverse value of the required failure rate in this example.

It is thus obvious that any particular fleet or part of all the airplanes belonging to a class or category of planes, for which a certain maximum failure rate is prescribed, will have to be designed to a level of probability of failure which is normally very much less than that at which one plane of the fleet would be likely to fail.

The relationship between P and F is furthermore exemplified by Fig. 7, which in principle gives a P-F-S_{max}-T-diagram for constant amplitude loading typical for pressure cabins—i.e., with the lower load limit equal to zero. This figure illustrates the hypothetical case where only the scatter in the S-N qualities of the structure has to be taken into account and where no redundancy or possibility of preventing failure by inspection and repair exists.

3.6. *The Probability Factor*

The proposal that the failure rate for catastrophic failures should be the basis of the fatigue safety concept, and the fact that this failure rate must be kept extremely low, makes the corresponding limit probability of failure so small that it might be difficult to assess by statistical methods. The limit probability, however, will only have to be applied in the design of the various elements of the airplane when there is *no* chance of preventing a fatal failure by discovering the fatigue crack in time and when the structure is statically determinate, implying that failure of one element causes failure of the whole structure. This, however, must be regarded as a hypothetical case. The designer will mostly be concerned with the probability of failure in the individual fatigue sensitive elements or parts of the airplane. This probability level, which is called the element probability, P_E, is normally very much higher than the limit probability, thus

$$P_E = c_P P_L \qquad (14)$$

where c_P, "the probability factor," is introduced as a measure of the ratio between the probability, P_E, of fatigue failure of the most fatigue sensitive vital part and the probability of a catastrophic accident. The probability factor can be split up into two main components

$$c_P = c_{Pi} c_{Pd} \qquad (15)$$

of which c_{Pi} is the increase in the probability level due to inspection and c_{Pd} is the increase due to the type of design used, in particular the inherent degree of redundancy.

The implication of the probability factor and its two components will now be briefly discussed with reference to the schematic Figs. 8 and 9 illustrating the conditions of a statically determinate system and of a highly redundant system, respectively. These figures illustrate a number of distribution curves, the two most interesting being the "element curve" (0), for fatigue sensitive elements of the structure, in particular for the most sensitive element, and the "critical curve" (6),

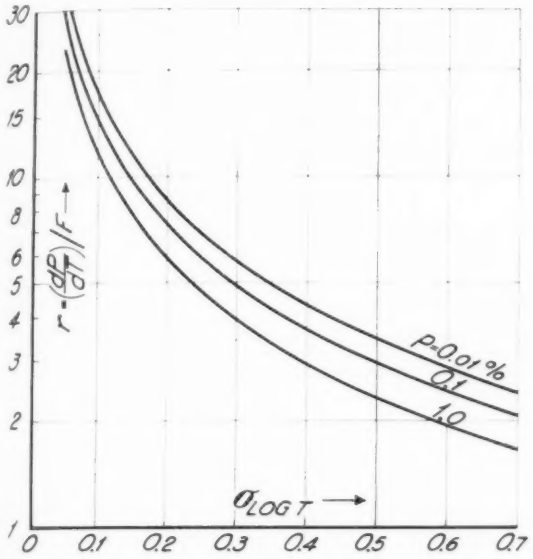


Fig. 6. Ratio between risk of exposure and failure rate versus standard deviation in log T at different critical P-values.

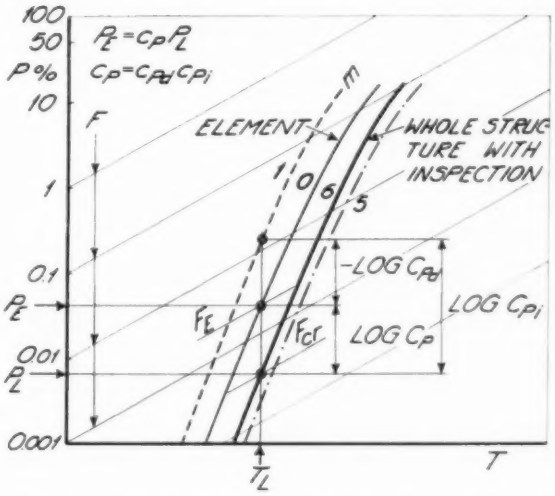


Fig. 8. Distribution curves for element and complete statically determinate system, without and with inspection.

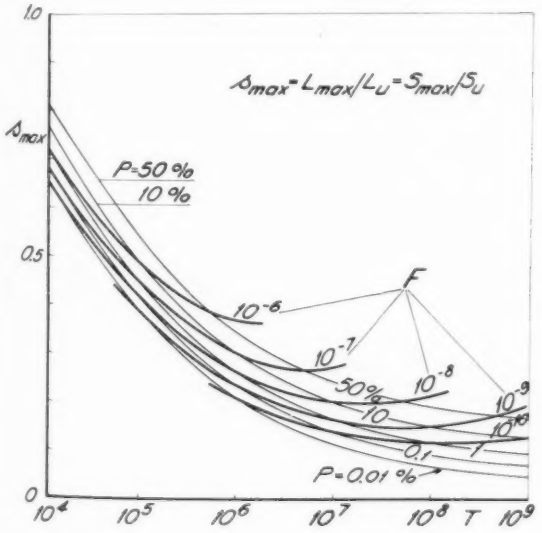


Fig. 7. Diagram showing relative maximum load versus life for different probabilities and failure rates, typical for pressure cabins.

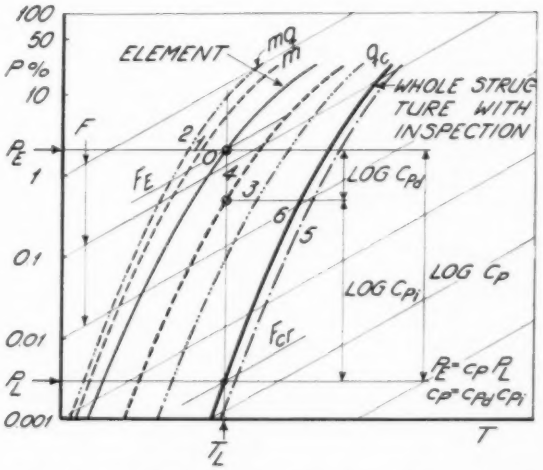


Fig. 9. Distribution curves for element, component and complete highly redundant system, without and with inspection.

for failure of the entire structure. Although the designer is interested primarily in obtaining the element curve from the critical curve, which would give him the element probability for which the elements should be designed, the curves will be explained in the opposite order for sake of simplicity.

In Fig. 8 the element curve (0), which passes through the point T_L, P_E, is the distribution curve for the most fatigue sensitive element. This curve is assumed to have been determined by the use of either information regarding the *P-S-N*-qualities of structural elements similar to those intended to be used in the design or corresponding information from varying amplitude tests. From the element curve the curve (1) is calculated by assuming, for instance, that the structure consists of m vital elements each with an equal or lower nominal fatigue sensitivity than that for the element represented by curve (0). Curve (1) is thus the distribution curve corresponding to complete failure of the entire structure as the redundancy is assumed to be zero. This curve is valid for the imaginary case where failure of an element is never prevented by detecting cracks during inspection. In practice, however, catastrophic failures might be precluded by inspection even in the case of statically determinate structures, due to the fact (see Section 3.1) that an appreciable and also detectable crack can usually occur before the static strength is reduced to such an extent that the structure cannot withstand the manoeuvring or gust loads likely to occur during the flight preceding the detection of the crack. How much the distribution curve for the complete structure can be improved or lowered, due to inspection is, of course, dependent on a number of factors, such as the rate of crack propagation, the extent to which cracks can develop and grow to critical depths without being possible to detect (mostly in the interior of heavy joints), and above all, the length of the inspection intervals and the thoroughness of these inspections. The assumption of a 100 per cent effective inspection carried out after each flight would lead to a rather substantial lowering of the probability of failure of the whole structure, curve (5). In reality, however, one must take into account the possibility that some cracks will pass undetected through one or a number of inspections, which results in the realistic final probability curve (6) for the whole structure. This curve should pass through or below the point T_L, P_L.

Turning now to Fig. 9, a redundant structure is assumed to consist of q main components—e.g., spars, each comprising m vital elements in series. Curve (2) is the distribution curve for failure in any of the mq elements. It is further assumed that the whole structure does not fail until q_c components have failed. As a first approach, it can be assumed that failure of any one of the components does not imply a change of the load in the other components, curve (3). This, however, would practically always be an erroneous assumption and on the unsafe side, as a quite substantial increase in the loads of the other components, particularly in those adjoining the fractured component, must normally be taken into account. If this is considered, curve (4) is obtained. This curve thus represents the distribution curve for the case where no inspection at all is carried out. A 100 per cent efficient inspection after each flight would lead to a very low theoretical curve (5), whereas a realistic assumption that on the average only a certain percentage of cracks or part failures will be detected at inspections often at longer intervals than after each flight, would give the critical curve (6), which again should pass through or below the point T_L, P_L.

It might be pointed out that redundant structures can be built up in a vast number of different ways and that the above description will therefore only be representative for certain cases. It will be the task of the designer to analyse his particular type of structure in order to assess the relevant distribution curves, or in some other appropriate way, calculate or estimate the total probability factor c_P.

The above description of various distribution curves for the cases of statically determinate and redundant systems has not covered all the uncertainties prevailing in aeronautical fatigue. A few words might therefore be mentioned about the other uncertainties briefly described in Section 3.3. In the first place it should be pointed out that in establishing the "element curve" (0), consideration has to be taken to any difference in tolerances between the fatigue sensitive elements of the airplane and the specimens used for the applied design information. The fact that elements in a redundant structure might be loaded more corresponding to constant strain than to constant load or stress, is deemed to be a slight relief for an element in which a crack is propagating. It therefore normally should be safe to neglect this matter in spite of the fact that the relief in the loading of the cracking element causes an increase in the loading of other elements, in particular the adjoining ones. Furthermore it might be emphasized that the element curve naturally must be based on static analyses of the structure giving the load level in the element. Any uncertainties in this respect should be covered by conservative assumptions.

The uncertainties connected with cumulative damage are of different magnitude and nature if the design information is based on *S-N* testing or on varying amplitude testing. In the former case the "element curve" has to be calculated with due consideration to the relationship between the scatter in *S-N* results and at varying amplitude loading, a matter which calls for special research. As regards the uncertainty in the cumulative damage theory itself it might preliminarily be proposed that, inasmuch as this uncertainty mostly seems to be a matter of ignorance, the uncertainty should be taken into account by a conservative assumption as regards the applied theory, for instance by putting the critical sum of the part damages somewhat lesser than 1.0. In case the element curve is based on varying amplitude testing the said uncertainties are eliminated in principle, but there is still the remaining uncertainty that some very high loads (possibly ex-

cluded in the loading program for the testing) might contribute markedly to the damage and in a different way, depending on when such high loads occur during the lifetime. This matter could be taken into account by considering the part damage due to such high loads. For sake of simplicity it might be assumed that such loads occur at stages of the life when they cause the most damage, which in most cases might be toward the end of the fatigue life. A more complete and partly statistical treatment of this uncertainty might also be possible.

The uncertainty as regards the number of applied loads (as apart from the composition of the load spectrum at a given total number of loads) is to such a high degree dependent on the nature and number of the fatigue meters used in the individual airplane, that no general rule can be given for a statistical treatment of this matter. Therefore the only rational way of dealing with this question is to exclude this uncertainty from the fatigue life and fatigue probability calculation by basing all the calculations on an assumed "basic" load spectrum curve, as will be discussed in Chapter 5. Any deviation in actual service from this basic assumption has to be accounted for in connection with the operation of the individual airplane. In doing so, due consideration has to be given to the uncertainty of the fatigue metering equipment used as regards the accuracy with which the equipment is capable of recording the load history of the airplane or the damage effect of actually applied loads.

Finally it might be pointed out that it is of course impossible to give any generally applicable numerical values of c_P, or of any of its parts. In general it might be stated, however, that both c_{Pd} and c_{Pl} are considerably larger in a redundant system than in a statically determinate one. It might also be possible to design a structure in such a way that, even if the elements are designed to a rather high probability level, the probability of fatal failure for the whole plane, would fall below the value corresponding to a certain required upper limit of the critical failure rate. Alternatively, it might be possible to extend the life of such a structure beyond the required limit life, so that a prescribed maximum failure rate is just complied with. The price to be paid for the gain in failure rate or in life is obviously the necessity to carry out a considerable number of repairs including even replacements of major components. It seems quite possible that the "repair rate" could become prohibitively high and thus an economical limit would be set on the level of the acceptable probability of failure for the individual elements, implying a shorter life than that corresponding to a prescribed maximum failure rate for the whole airplane.

3.7. *Summing Up the Problem*

After this general review of the most important aspects of the aeronautical fatigue problem we are in a position to sum up the problem as it confronts the designer and the operator, and to outline briefly the kind of research work that has to be carried out.

The *designer's* problem can suitably be divided into the following two topics:

Topic A. Designing for fatigue.

It is nowadays rather widely agreed that the fatigue problem should to the greatest possible extent be dealt with already in the design stage of a new type of airplane. During this stage, however, no parts of the airplane of the ultimate shape are available for fatigue testing and the designer will therefore have to resort to general kinds of fatigue information applicable to the intended type of design. This consequently implies that one of the most important aims for research in aeronautical fatigue is to establish reliable design information, and convenient methods of utilizing it, enabling the designer to produce efficient structural designs which comply with certain specified fatigue qualities, be they specified by the designer himself or in official requirements. Such qualities should primarily be expressed in terms of failure rate. For the case of highly redundant and easily inspectable types of design additional criteria in terms of repair rate might have to be established, preferably in cooperation with the operators.

Regarding the type of fatigue testing that should preferably be used in order to establish generally applicable design information, the following might be said. Varying amplitude tests have the advantage of giving direct information as regards the fatigue life of a specimen under varying load amplitudes, which implies that no more or less questionable cumulative damage theory needs to be applied. One disadvantage with this method is that the test results have a more limited value as they are only valid for the particular type of load spectrum applied in the tests, whereas the results of S-N tests are in principle applicable for any type of gust load spectrum. Furthermore, the equipment or test procedure needed for varying amplitude tests is more complicated or troublesome than for S-N testing. In view of these pros and cons it is believed that there is a need for a considerable extension of varying amplitude testing, both in order to establish directly applicable design information and for checking cumulative damage theories. The S-N type of testing, however, will most likely remain as an important basis for establishing generally applicable design information. As results from varying amplitude tests are very scarce, this paper deals mostly with the particular problems involved in transforming the results from S-N tests into directly applicable design information.

In view of the preceding Section, the designer's problem at the design stage can be schematically summarized as follows.

1. To analyze the structure in order to assess the element probability level, P_E, corresponding to the limit life time, T_L, for which the structural elements should be dimensioned, in order to comply with a desired or required maximum value of the critical failure rate (and possibly also repair rate).

2. To determine that load or stress level for the elements, expressed for instance in the fatigue factor, k_A, which gives a probability level lower or equal to P_E at the desired limit life.

Topic B. Checking fatigue properties.

When the structural design is "frozen", the designer has the possibility of checking the design, as regards its fatigue properties, by manufacturing and testing samples of vital and fatigue-sensitive parts of the airplane. At least with the present stage of knowledge such confirming tests are highly desirable inasmuch as the general fatigue information available to-day is rather inadequate as a basis for the design. It is believed, however, that even when the designs can be founded on much more extensive and reliable fatigue information of a general kind, check tests on vital parts will remain as a highly important criterion, at least for non-redundant structures.

For check-testing the variable amplitude method has some definite advantages, but as it involves a more expensive type of test equipment, constant amplitude testing at a limited number of representative load levels, possibly only one, is likely to play an important role for this purpose also in the future.

As the vital, fatigue-sensitive parts are mostly very expensive, it is normally not possible to carry out check-testing with more than rather few samples. It is therefore an important research topic to establish statistical methods by means of which the fatigue qualities of a new type of airplane can be determined as accurately as possible from the results of a few check-tests if possible in combination with how and to what extent the type has been designed for fatigue. In case *S-N* testing has to be made, a special research subject is to establish methods for determining the most efficient number and magnitude of load levels that should be applied to get the most out of a limited sample size.

Check tests can evidently also be performed with complete components of the airplane, such as a wing or a pressure cabin. For economical reasons it is even less possible to carry out such tests with so many components as would be desirable from a statistical point of view. The main value of such full-scale tests of large components seems therefore to lie in the possibility they offer of revealing fatigue-weak regions or parts, other than those which were considered in the design stage. Another important object of testing complete components of redundant structures is to determine the redistribution of the load pattern when any one member has been cut off simulating fatigue failures in different parts of the structure.

Also the *operator's* part of the fatigue problem can be divided into two main topics:

Topic C. Fatigue metering.

As has been emphasized above there is one quite efficient way of considerably reducing the total amount of uncertainty in aeronautical fatigue and that is by means of recording either the load history for the individual airplane or the damage effect due to the repeated loads. Several devices have been proposed to meet this demand and it is believed that extended research and development work regarding all feasible kinds of fatigue metering is of utmost importance. A related research topic is of course to establish reliable methods for rapid evaluation of results recorded with meters.

Topic D. Inspection and repair.

Careful inspection with efficient methods in order to detect fatigue cracks at as early stages as possible, will always be necessary, although it is hoped that for future types of airplanes, it will not form the main basis for safety in respect of fatigue. As regards older types of airplanes, in the design of which fatigue has not been a prime consideration, inspection is of course of special significance.

The question of the repair rate and in particular its increase with service life is of great economical importance for the operator. Although the repair rate depends on the design, the economic consequences have to be taken by the operator, and for this reason the furthering of studies of this matter is primarily of interest to the operators.

The matters of inspection and repair will be dealt with only rather briefly in the following, as they fall a bit outside the main scope of this lecture.

4. Level of Safety

4.1. General

After this survey in general terms of the aeronautical fatigue problem, its main aspects will be treated quantitatively. This should logically start with a discussion of the level of safety to be aimed at in terms of numerical values of the maximum permissible failure rate for catastrophic failures. The discussion will be limited to commercial transport airplanes.

The level of safety in respect to fatigue should naturally be seen against the background of the overall level of safety for commercial aviation. In this respect there are two circumstances, or commonly held opinions, that are worth emphasizing.

One is the fact that the level of safety in civil air transportation has undergone a quite remarkable and steady improvement during the last twenty years, and that it has now reached a level which would seem to be satisfactory for any passenger with reasonable demands on a practically risk-free existence. The number of passenger fatalities per 100 million passenger-miles for the last two years (1952 and 1953) was only of the order of 1.5 on the world's scheduled air services, and for the U.S.A. the corresponding figure was as low as around 0.5.

The other fact is that, good as these values of the level of safety in air transportation are, they are still not satisfyingly low in comparison with commercial surface transportation, in the first place by railroad. During the years 1950–1952 the rate of fatal passenger accidents counted in fatalities per 100 million passenger-

miles was, in the U.S.A., about 2.5 times as high in the air as on the railroads, and in Europe the ratio was as high as 6.3. This does not seem too unsatisfactory, but as Tye has pointed out in reference 12, the passenger is probably more interested in the risk per journey, and assuming the average air trip to be about 10 times as long as the typical train journey, the risk per journey would be considerably greater in the air than in the train. This, in combination with the fact that the chance of survival in case anything serious happens is much less in an airplane than in a train, seems to justify a mostly subconscious but probably rather common feeling on the part of the general public, that commercial surface transport is still considerably safer than commercial flying. Although this opinion, in view of the excellent safety record obtained in air transport, particularly in the U.S., seems exaggerated, it must be considered as an important reality, because public opinion is the very foundation for the profitableness and expansion of commercial aviation, regardless of whether or not the opinion is justified by statistical facts.

At this point it might be worth while to dwell a little on the question of level of safety versus economy. Now and then one encounters a more or less vaguely expressed opinion that an increase in the safety of air transportation, were it possible to accomplish, to a level at which practically no fatal accidents would occur, or even to the same level as that of the railroads, would result in a prohibitive increase in the cost for transport by air, due to reduced payload, and increase in the cost for the ground organization. This kind of thinking seems to be based on a feeling that there would be a contradiction between safety and economy in air transportation, implying that the problem would be to find an acceptable compromise satisfying both demands. The author believes that such thinking is not only dangerous, but incorrect, as it is difficult to imagine any more efficient way to improve the profitableness of aviation than with a marked raising of the safety level. Of the millions of people flying to-day, a considerable number, possibly the majority, hardly give the risk of flying a thought, but no doubt a substantial part of the passengers balance the gain in time against a feeling of risk. There are also undoubtedly millions of potential passengers who never fly at all, due to this same feeling. A safety record equal to, or better than, the railroads' would most certainly result in an enormously increased demand for air transportation, and the only limitation to meet it would be the pace with which the facilities in airplanes, ground equipment and personnel could be expanded. Briefly stated it is believed that there is no better way to improve economy than to transport more people by air, even if improved safety —for instance in respect to fatigue—would necessitate a certain increase in structure weight —i.e., decrease in payload.

Thus safety in aviation should be put in a class by itself far above all the economic considerations in the limited sense of "block-to-block economy," and the author is convinced that this is being done by most people seriously engaged in the safety problem. There might, however, be some differences in the views as to the best policy to follow in order to improve still further the safety record in civil aviation. A natural, and in a way rather sound, approach is to concentrate on those accident causes, which are responsible for the most common of the serious occurrences and devote somewhat less attention to the less frequent causes of accidents. For instances, errors made by pilots and other personnel seem to be responsible for about 50–75 per cent of all the accidents, whereas lack of airworthiness only causes some 10–25 per cent, and of this latter category only a small fraction is caused by structural deficiency.[12] The said trend of opinion would consequently imply a concentration on attacking the operational causes and a feeling that safeguarding the structural airworthiness would be less important. The author believes, however, that this "rule of proportionality" should be somewhat modified by applying a "scale of excusability" and this opinion is based on the observation that the public, more or less subconsciously, does discriminate between different causes of accidents. To exemplify this from the field of transportation by private cars, everyone knows that collisions between cars do happen and that they are probably impossible to prevent altogether, and it is furthermore commonly accepted that engines do stop now and then and that some trouble with mechanical pieces of equipment will always be inevitable. The reaction of the car-owners would certainly be of quite another and much more severe nature if structural failures of automobiles became a rather common source of fatal accidents.

Likewise, the public might feel that some causes of air accidents, such as a collision with a private plane flying carelessly, failures of engines and other pieces of machinery, extremely bad weather conditions and some types of human error, are, if not tolerable, at least understandable. Failure of the primary structure itself, however, is probably regarded as inexcusable, in the first place, because the public might rightly think that it should be possible to design a non-moving construction so that it can withstand all loads occurring in commercial flying, and secondly, because a failure of the main structure, for instance the wing or the pressurized cabin does not normally give the passengers any chance of survival in contrast with most other serious types of air accidents.

In the author's opinion there should moreover rightly be some differentiation between static and fatigue failures, in respect of their degree of excusability. To encounter an excessively severe gust, for instance by running into a non-visible jet stream, *can* happen, without being possible to prevent or being anyone's fault, although the probability of such occurrences is so utterly remote that it ought to be ignored by the public. A total collapse of the wing due to fatigue damage when flying in fine and moderately bumpy weather, approaches the kind of happenings which are not excusable under any circumstances.

In conclusion it might be said that when, as at present, failures of the main structure quite naturally constitute only a very small part of the accidents, it is vitally important not only to maintain this proportion, but also to work energetically in order to reduce the occurrences of such failures, and in particular of fatigue failures, to a level where they are practically unheard of.

4.2. *The Acceptable Failure Rate*

If the views expressed above are accepted, it would seem that the target to aim at regarding catastrophic fatigue failures would be 100 per cent safety—i.e., the probability of failure, and the failure rate, would have to be zero for all types of commercial transport airplanes. To limit the probability level to zero, however, is not necessary in order to achieve the goal of making fatal fatigue failures "practically unheard of," and it certainly would make the statistical treatment of the related problems rather difficult. As furthermore aiming at zero probability of fatigue failure would result in very heavy structures, it seems sufficient to decide upon a very low, but finite figure for an acceptable failure rate. To fix this value is an important matter of such a nature that it should preferably be handled by the various National Authorities, and by ICAO. As a preliminary suggestion the author feels that the acceptable failure rate should be fixed at such a value that no more than one catastrophic failure can be expected during one or two generations—i.e., in a period of about 30–60 years.

In 1953 all the world's scheduled services available for statistical records, reached a total of 1,170 million plane-miles flown.[13] Assuming the average block speed to be around 200 m.p.h., this would correspond to about 6 million flight hours. During the last three years the increase in miles flown was around 10 per cent, whereas the average block speed seems to have been increased by about 4 per cent per year, which would mean that the increase in flight hours has been approximately 6 p r cent per year during the last few years. If we now try to look into the future for a period of about one or two generations, and if we dare to assume that the annual increase in commercial aviation, calculated in hours of flight, will be maintained at about 6 per cent, we will find that from now on a total number of 10^9 hours will have been completed in slightly more than 40 years. The critical failure rate is thus preliminarily suggested to be fixed at

$$F_{cr} = 10^{-9}$$

This value might possibly seem to be unduly low, but it should then be pointed out that working to this failure rate cannot be regarded as a guarantee that not more than one failure will occur in 10^9 hours of flight. Statistically seen, the said failure rate only implies that if a considerable multitude of 10^9 hours were flown, the number of failures that could most likely be expected would be 10^{-9} times the total flying time. In 10^9 hours the number of failures might be zero, but it might also be 1, 2 or 3, or even more. If we really wish to accomplish the task of practically preventing catastrophic fatigue failures, the author maintains that the above value of F_{cr} is of the correct order of magnitude. As this entire topic, however, will probably be an unsettled question for some time to come, it might be pointed out that the equations and methods presented in the following are of a general nature, and that any other numerical value of the critical failure rate can be applied.

5. The Gust Loads

5.1. *The Gust Load Equation*

The well known equation for the acceleration increment due to a gust is

$$\Delta n = \frac{1.47\ \rho_0\, a K V_e U}{2\ W/S} \tag{16}$$

with symbols as given in Chapter 2.

Due to the obvious fact that this equation is valid both for the static gust load case and for any current gust, certain rather simple relationships can be derived if the loads or stresses caused by any gust are related to those loads or stresses, which correspond to the prescribed "limit" gust of 50 ft./sec.* Such a procedure, however, only seems to be of significance when the limit gust load factor is equal to or greater than the limit manoeuvring load factor, which does not always apply. Furthermore, the said procedure seems to render more difficult a general treatment of the effects of over-dimensioning of a structural element as well as of the conditions when speed and weight differ from the design values.

To obtain a clearer picture it seems preferable to relate the varying loads, due to gust, to the maximum ultimate static load factor, whichever load case determines its value. The gust sensitivity of the airplane can obviously still be expressed as the gust load factor. In order to have to deal only with one load factor, the maximum one, and thus avoid confusion, it seems preferable to express the gust sensitivity in a relative way. This can be done by defining the "relative gust sensitivity factor," ν, as the ratio between the acceleration increment in the static gust load case ($U = 50$ ft./sec.), for the airplane considered and that value (1.5) for this increment, which applies for transport airplanes having such a gust sensitivity that the ultimate gust load factor just reaches the value, 3.75.

With this approach, the following equations are easily obtained:

$$s_a = \frac{L_a}{L_u} = \frac{S_a}{S_u} = 0.03\nu \frac{v}{n_D k_A \omega_a} U \tag{17}$$

$$s_m = L_m/L_u = S_m/S_u = \omega_m/n_D k_A \tag{18}$$

* This value of the limit gust is in accordance with the ICAO requirements.[14] In Civil Air Regulations[15] the limit gust is prescribed to be 30 ft./sec. but the alleviation factors are defined in different ways so that the two requirements are virtually the same.

where

s_a = relative load or stress amplitude
L_a = absolute load amplitude
L_u = absolute ultimate strength of element
S_a = absolute stress amplitude in element
S_u = ultimate stress in element at failure
ν = relative gust sensitivity as defined above
n_D = ultimate design load factor in the most critical static load case
k_A = "fatigue factor," which at the same time is a measure of the static margin of safety. (subscript A denotes an increase in area)
$\omega_a = f(\omega)$, factor representing the effect on s_a of a weight reduction, expressed by $\omega = W/W_D$, the ratio of actual airplane weight to design weight
v = V_e/V_{eD}, relative equivalent airspeed
U = equivalent vertical gust velocity, ft./sec.
s_m = relative mean load or stress
L_m = mean absolute load or 1g load
S_m = mean absolute stress or 1g stress
$\omega_m = f(\omega)$, factor representing the effect on s_m of a weight reduction, ω.

The relative gust sensitivity can be calculated from

$$\nu = \frac{1.47\ \rho_0 a K V_{eD}}{0.03 \times 2\ W_D/S} \tag{19}$$

where

ρ_0 = standard air density at sea level, slugs/cu. ft.
a = slope of wing lift curve, per radian
K = alleviation factor
S = wing area, sq.ft.

The gust alleviating factor, K, in Eq. (19) might mean either F, as defined in the ICAO-standards,[14] or K_g as defined in reference 16. In both cases K is a function of the wing loading, but the variation in K due to reduction in all-up weight for a particular airplane can usually be considered as negligible.

It is sometimes of interest to introduce the concept of the "element load factor"[17]

$$n_E = L_u/L_m = k_A n_D/\omega_m = 1\ s_m \tag{20}$$

In case the fatigue factor, k_A, as well as the factor ω_m are 1.0

$$n_{E1} = n_D = L_{uD}/L_m \tag{21}$$

where L_{uD} is the required ultimate design strength in the critical static load case.

The effect of a weight reduction on s_a and s_m is dependent on a number of parameters, such as the original relative wing weight, μ, the distribution of the wing weight in relation to the wing lift distribution, how the weight reduction is distributed between wing and the rest of the airplane and the location of the element under consideration.

In Figs. 10 and 11, ω_a and ω_m are presented for a structural element located close to the fuselage on the tension side of the wing. The curves are valid for a few typical values of the important parameters, as explained in Fig. 10.

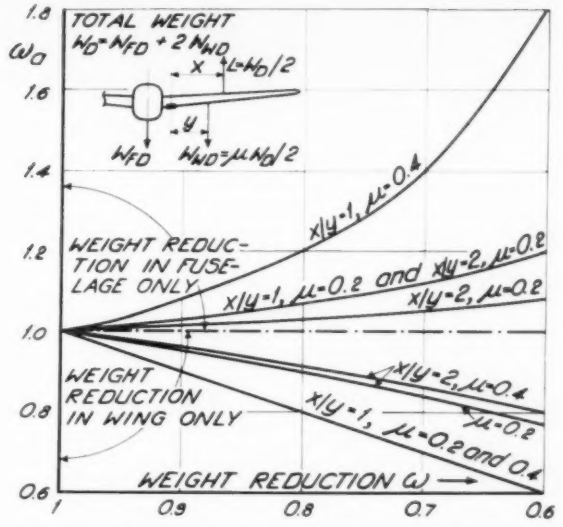


Fig. 10. Effect on ω_a of weight reduction. The chain-dotted abscissa represents the case when the weight is reduced both in the wing and the fuselage and in the same proportion as the original ratio between the wing and the fuselage weight.

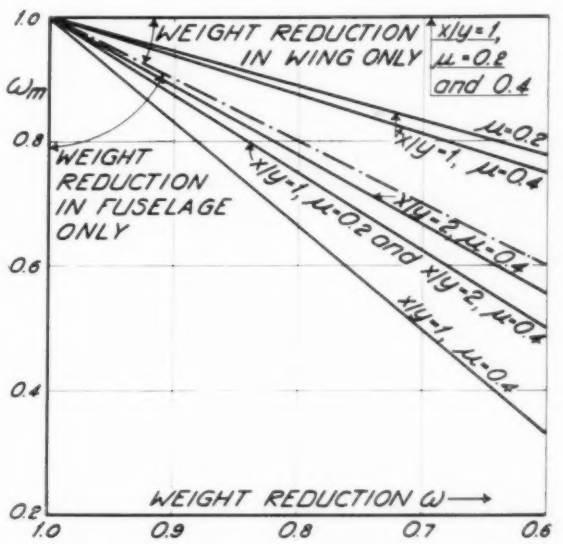


Fig. 11. Effect on ω_m of weight reduction. The chain-dotted line represents the case when the weight is reduced both in the wing and the fuselage and in the same proportion as the original ratio between the wing and the fuselage weight.

It should be pointed out that the above equations are only valid for the "rigid body" case. The change in the loads and stresses due to dynamic response can be quite appreciable.[18,19] As, however, the dynamic response can hardly be treated quantitatively in a general way, it will be the designer's task to account for

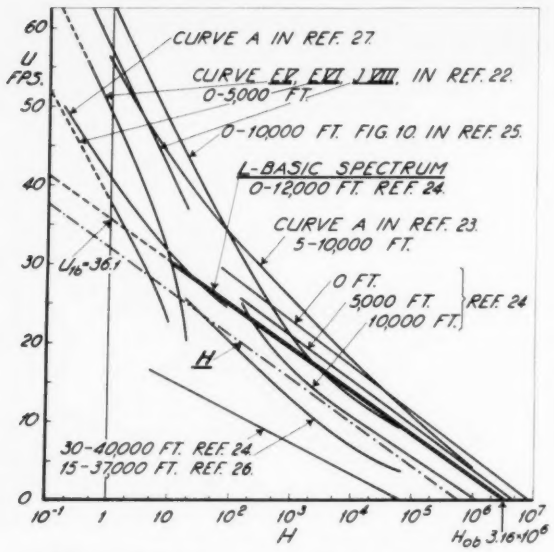


Fig. 12. Gust spectrum curves from various sources, referred to a distance of 10^6 miles.

any significant deviation between the basic rigid body case considered above and the dynamic conditions.

The concept of the relative load amplitude, s_a, the relative mean load, s_m, and the fatigue factor, k_A, are significant and practical to use in the design stage, "Topic A," when the structural element has to be designed so that it meets both static strength requirements and desired fatigue qualities. When, however, a structural element or component, after design and manufacture has to be checked in respect of fatigue, "Topic B," its ultimate strength, might be of a somewhat secondary interest, at least when it has been found at the design that a large fatigue factor, k_A, implying a large static margin of safety, is necessary to apply in order to attain the limit life aimed at. Thus the plotting of the test results to determine the endurance curve might just as well be done in absolute values—e.g., in tons or in stress, defined in one way or another. It might, however, be of some value to relate the load amplitudes to the mean load applied in the tests, because this will widen the applicability of the endurance curves for general treatments. This defines the "comparative load amplitude," which from Eqs. (17) and (18) can be expressed in the form

$$l_a = \frac{L_a}{L_m} = \frac{s_a}{s_m} = 0.03\nu \frac{v}{\omega_a \omega_m} U \qquad (22)$$

An advantage with this parameter is that the number of variables has been reduced, as the fatigue factor, k_A, and the ultimate design factor, n_D, have vanished.

5.2. *The Gust Load Spectrum*

During the last two decades gust load data have been accumulated at an ever-increasing rate. At present a considerable number of transport and other types of airplanes are equipped with recording equipment of various kinds in order to extend the knowledge of the turbulence of the air as well as the gust intensity spectrum on different routes and operational conditions. The results obtained have been published in a considerable number of reports and have been reviewed in comprehensive articles.[20–22] Therefore this side of the aeronautical fatigue problem will only be dealt with to the extent necessary for meeting the purpose of this paper.

In Fig. 12 some gust spectra in the form of summation curves have been selected from various sources[22–27] in order to obtain an overall picture of the general appearance of the curves, their variation with altitude and their spread due also to other reasons. The figure shows the gust spectra in the coordinates U ft./sec. and log H, H being the number of gust cycles, in 10^6 miles, equal to or greater than U. From this figure it can be seen that the total spread in the results is of the order of a factor of nearly 20 for altitudes up to 10,000 ft. and about 1,000 if the curve for altitudes above 30,000 ft. is taken into account. The only practical way of introducing gust load data in the fatigue picture is obviously to define a "basic gust spectrum curve" or function which is fairly representative for certain overall conditions. Once such a basic spectrum has been determined, a second problem is to evaluate the effect of any, predicted or measured, deviation from the basic spectrum in actual flight.

Taylor[24] has presented an average gust spectrum curve for airplanes flying up to 12,000 ft., a curve which has also been used by Chilver.[28] This spectrum, which is a straight line in U vs. log H coordinates, will be the "basic" spectrum in this paper, on which the numerical calculations are being based. The curve is marked "L" (low altitude) in Fig. 12. Any straight line is defined by the two intersections with the axes (1, U_1) and (H_0, 0). For the basic spectrum these points are (1, U_{1b} = 36.1) and ($H_{0b} = 3.16 \times 10^6$, 0).

The assumption of a straight line relationship in U vs. log H might seem to be unduly on the unsafe side as most of the spectrum curves are bent upwards very markedly at higher gusts. It is believed, however, that the error introduced by using the straight line is negligible, in the first place because the relative fatigue damage due to the so rarely occurring gusts above around 25 ft./sec. is usually very small. It might also be said that it seems probable that progress in meteorological knowledge and instrumentation enabling still better planning of the operations in order to avoid regions of highly turbulent air, might lead to a flattening out of the upper parts of the curves making the slope of the L-line more representative in this range.

Regarding the lower end of the basic curve, or any other straight line, it might be mentioned that this part cannot represent accurately the real conditions in the atmosphere, as the number of extremely small gusts is a rather undeterminable matter. The intersection of the L-line with the abscissa at H_{0b} should therefore be understood as a point for the definition of the line.

Any straight gust spectrum line in U vs. log H can be transformed to a corresponding basic gust load spec-

trum by using Eq. (17) for the determination of the relative load amplitude, s_{a1}, corresponding to U_1, and by applying the H_0 value valid for the gust spectrum (see Fig. 2). Expressed in a form convenient for the application of the cumulative damage theory, the equation for the gust load spectrum is

$$\log H = \log H_0 - s_a h \log e \tag{23a}$$

or

$$H = H_0 e^{-h s_a} \tag{23b}$$

where H_0 is equal to H_{0b} for the basic gust curve, and h, the slope of the line, is

$$h = \frac{n_D k_A \omega_a}{0.03\ \nu v \log e} \frac{\log H_0}{U_1} \tag{24}$$

For the basic gust spectrum and with $n_D = 3.75$ and k_A, ω_a, ν and $v = 1.0$ the slope is

$$h_{b0} = 51.8 \tag{25}$$

For any gust spectrum curve parallel with the basic curve, the slope h for the load curve can be expressed in the form

$$h_h = 13.8 \frac{n_D k_A \omega_a}{\nu v} = 51.8\, k_{Af} \tag{26}$$

where k_{Af}, a fictive value of the fatigue factor, is

$$k_{Af} = \frac{n_D \omega_a}{3.75\ \nu v} k_A$$

In order to predict the change in gust spectrum, and gust load spectrum, for operations at higher altitudes, the ascending and descending stages of the flight must be taken into account. In Fig. 12 the curve H has been calculated under the assumption that the operational height is 30,000 ft. and that, on the average, one-third of the distance is covered during the said two stages together. It was found that this curve, H, is very nearly parallel to the basic curve and is displaced a factor of about 6 in H.

6. *P-S-N*-Information

6.1. *General*

As pointed out above an important part of the general information needed by the designer must consist of, or rather be based on, results from *S-N* tests. In order to be *directly* applicable for this purpose, *S-N* tests must comply with the following conditions:

1. The sample size must be so large that the results can be treated statistically so as to enable the designer to design the structural elements for the required life corresponding to a certain probability level of the order of 0.01 to 1 per cent, this P_E-level being dependent mostly on the intended type of construction. A strict application of this condition would mean that for each load level the sample size must have a magnitude of about the inverse value of P.

2. The test specimens must be made of material of the same specification as for the material to be used in the actual structure.

3. The tests must be carried out under practically the same loading conditions as are typical for the actual structure in service, with the exception, of course, that constant amplitude loading implies a deviation from the actual type of loading. This means for instance that for applications to wing structures the tests should be run with a series of constant mean loads of representative orders of magnitude. The relative mean load, s_m, should thus be within the approximate limits 0.27 and 0.135, corresponding to k_A values between 1.0 and 2.0 and $n_D = 3.75$.

For pressurized cabins the tests should normally have the lower load limit, s_{min}, equal to zero.

4. The test specimens should, irrespective of their size, be geometrically representative of the elements intended for the design, so that there are no appreciable differences in stress concentrations.

5. The test specimens should not differ too much in size from the actual structural elements.

6. The sample of test specimens should be manufactured to tolerances very similar to those which are worked to in the normal production of the structural component in question.

Although a tremendous amount of *S-N* testing has been carried out in the century-old history of fatigue research, there are, to the knowledge of the author, extremely few, if any, published test results which comply with all these conditions even for rather conventional types of design. There is therefore an urgent need for extensive *S-N* testing of specimens, which should be as representative as possible of the types of design likely to be used by designers. It must, however, be borne in mind that there is a practically infinite number of conceivable design principles and detail shapes and sizes of elements, applicable for instance to wing structures. Consequently it will probably never be possible to accumulate *S-N* results to such an extent that a designer normally will be able to base a new design directly on available information of an adequate nature. In other words, regardless of how much *S-N* testing is carried out, the designer in most cases will have to apply some kind of methods to interpret or transform the available test results so that they can be applied to a particular design. A substantial part of the research efforts should therefore be directed to finding means for evaluating deviations in respect of the six viewpoints listed above, between available test results and actual conditions for predictable types of aircraft structures. This means that such factors as the influence of the mean stress or other representative loading parameters, the notch effect or stress concentration and the size effect, should be subjected to extensive investigations.

In general it can be stated that the more we know about these effects in a quantitative or, if possible, in an analytically graspable way, the more will it be possible to reduce the number of types of specimens with which

endurance tests have to be carried out—i.e., the more will the applicability increase of any series of S-N tests. It should, on the other hand, be emphasized that the uncertainty rapidly increases with the magnitude of the deviations between the experiments and their applications in design. The safest and most profitable way is therefore to conduct the experimental research on specimens with stress concentration factors and other geometrical or structural peculiarities of such orders of magnitude or detail shape, respectively, that the designer can in most cases apply the information by means of interpolation rather than by extrapolation. This in the first place applies to notch effects, the influence of which, to a large extent, should be studied by testing specimens representative of structural elements as used in practice. There is, of course, also a need for extended research aimed at relating the notch effect to the qualities of the unnotched material. Although such a more "basic" approach might be profitable in the long run, it has the disadvantage that the scatter in fatigue results is often found to be greater in unnotched than in notched specimens. This means that the basis itself, applicable for a particular design from which values should be extrapolated, by means of some empirical method, would be rather uncertain, which implies, for one thing, a great difficulty to determine the scatter in the fatigue qualities of a notched element, to be designed, from the (still greater) scatter found in the test results for the unnotched specimens.

In respect of the size of the specimens, however, the main part of tests intended for design information will probably have to be carried out with smaller sizes than those normally of interest to the designer. Consequently an extrapolation of the design information will usually be inevitable in this respect. This makes extended research on the size effect for various types of loadings and specimens necessary, but when more information is obtained it is probable that this source of error will be rather insignificant.

Regarding the necessary sample size it might be said that only in exceptional cases such large sample sizes as the inverse of the element probability—i.e., often 1,000 specimens or more, can be tested at each of a number of load levels, say from 5 to 10. A very important object of fatigue research is therefore establishing methods, in particular in order to find out to what degree and with what accuracy important statistical parameters can be assessed with a smaller sample size than that directly corresponding to a certain P-level.

In this connection it might be pointed out that it cannot be taken for granted that the scatter in the fatigue qualities of the actual elements in the structure under the real loading conditions is the same as the scatter in the S-N tests made in laboratories. The normally much greater tolerances for the elements in production airplanes would tend to increase the scatter, but on the other hand other factors might tend to decrease it, for instance higher stress concentrations. It is essential to account for all such factors. To find methods for this is another important research topic.

In order (1) to facilitate the establishment of methods for evaluating discrepancies between test results and all the actual conditions in the structure, and (2) to enable assessment of P-levels beyond those directly attainable in the tests, it is important to devote research work to finding relationships in the first place between the quantities P, S and N, where S represents the loading at constant amplitude tests. Whenever this loading is determined by two components, for instance s_a and s_m, the investigations should be extended to cover all the four variables—i.e., P, s_a, s_m and N for wing structures. Such investigations might preferably be split up into the two topics, studies of the S-N relationship, in particular for the mean values of test results, and studies of the scatter.

For the benefit of those who are not familiar with statistical concepts, a brief survey of some of the more important statistical definitions and equations will first be given.

6.2. *Some Statistical Concepts*

In order to enable a mathematical treatment of the scatter in fatigue tests, it is necessary to introduce a suitable distribution function. The most common one is the Gaussian, or the so-called normal distribution of the statistical variable x

$$P = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(x - x_{50})^2/2\sigma^2}\, dx \qquad (27)$$

where x_{50} is the median value (which is identical with the mean value at symmetrical distributions) and σ the standard deviation of x. The x-value corresponding to a probability level P is obtained from

$$x_P = x_{50} - U_P\sigma \qquad (28)$$

where U_P is the normal deviate, which is tabulated in most statistical textbooks.[10, 11]

In S-N-testing it has been found that a "log-normal" distribution of N usually fits the results with good accuracy—i.e., $x = \log N$. When the two parameters $(\log N)_{50}$ and σ are known, it is thus possible, by applying Eq. (28), to calculate the N-value corresponding to any P.

The main criticism against the log-normal distribution seems to be that the lower limit, $P = 0$, gives $N = 0$, whereas tests with large sample sizes show a tendency towards a lower limit $N > 0$. Partly in order to overcome this drawback Weibull[29, 30] has proposed the following distribution function

$$P = 1 - e^{[(N - N_u)/N_0]^{1/a}} \qquad (29)$$

where $N = N_u$ corresponds to $P = 0$. This function has three parameters, a, N_0 and N_u, which, of course, makes it more adaptable than the normal distribution. For small P-values ($P < 10$ per cent), the expression takes the simpler form

$$P = \left(\frac{N - N_u}{N_0}\right)^{1/a}$$

This function may also be used for approximating parts of the log-normal distribution with reasonable accuracy.

The advantages and disadvantages of the two types of distribution functions will not be discussed in this paper. It might suffice to state that they yield essentially the same results, unless an extremely low probability is required. In the following applications the log-normal distribution will be used unless otherwise stated.

As the sample size is never infinitely large, the distribution function should, strictly speaking, not be continuous. It may be approximated, however, by the above mentioned continuous functions. The standard deviation of a limited sample size is denoted by s, which will in this paper be given a suffix indicating the statistical variable considered.

$$s_x = \sqrt{\frac{\Sigma x^2 - (\Sigma x)^2/n}{n - 1}} \tag{30}$$

In some cases the coefficient of variation is of interest.

$$v_x = s_x/\bar{x} \tag{31}$$

Both the mean value and the standard deviation computed from a limited sample size are subjected to some errors. x_P-values calculated from the above Eq. (28) will also be uncertain. This uncertainty can be treated mathematically by introducing the concept of confidence, which will be briefly explained in the following.

As an example it is assumed that a small sample of n test specimens are chosen at random from a very large population and tested at a certain constant amplitude level. The mean value, $(\overline{\log N})$, and the standard deviation $s_{\log N}$ are computed. The same procedure is then repeated, at the same amplitude level, testing each time n new specimens, until the whole population has been investigated. After calculating the mean value of the whole population it will be found that the mean values of the small samples are in about 50 per cent of the cases higher than the true value. For one sample of n specimens the probability that the mean value, or the 50 per cent value, is too high, is thus 50 per cent. The same is true for values corresponding to $P = 10$, 1, 0.1 per cent, etc. The probability that only P per cent of the specimens fall below a computed value $(\log N)_P$ is commonly called "the confidence coefficient" and is denoted γ. Usually it is necessary to use a value of γ which is appreciably higher than 0.5 in order to be able to draw some conclusions, with a higher degree of certainty, from a particular test series which includes only a small number of specimens.

The concept of confidence is clearly applicable for any statistical variable. For the normal distribution the following procedure can be used for calculating x_P with a certain confidence γ, when $\bar{x}$ and s_x are obtained from the test series.[31, 32]

$$x_P = \bar{x} - k_P s_x \tag{32}$$

where

$$k_P = (U_P + \sqrt{U_P^2 - ab})/a \quad \text{(valid for } \gamma \geqq 0.5\text{)} \tag{33}$$
$$a = 1 - U_\gamma^2/2(n - 1)$$
$$b = U_P^2 - U_\gamma^2/n$$

U_P and U_γ are the normal deviates corresponding to P and γ. If $\gamma = 0.5$, $U_\gamma = 0$ and $k_P = U_P$.

In Figs. 13 and 14 the factor k_P is shown as a function of n for P-values from 0.0001 to 50 per cent and $\gamma = 0.90$ and 0.95, respectively.

If the true value of the standard deviation, σ, is known for any variable relevant to fatigue tests, such as log N, S or log S, Eq. (32) can still be used but the factor k_P will then be smaller. It would obviously be of considerable advantage if it, by means of accumulated experience, were found that the standard deviation in any of these variables could be predicted with a high degree of certainty for typical structural elements and types of loading. Schott[33] has recently treated the confidence problem in connection with fatigue tests under the assumption that the coefficient of variation, v, is known.

6.3. *The S-N-Function*

In this section the endurance or S-N-function will be treated without reference to any specific probability levels. A number of authors have suggested various equations for the endurance curve. One of the simplest, which as a rule represents test results with good accuracy has been applied by Weibull.[29] It can be expressed in the form:

$$N + B = \alpha(S - E)^{-\beta} \tag{34}$$

where

E is the endurance limit

B is a parameter, which must be taken into account for the left-hand, downward concave, part of the curve (the part to the left of the inflection point), but can be neglected for the right-hand, upward concave, part

α and β are parameters which must often be ascribed different values for the two parts of the curve. For the fatigue of airplane structures due to gust loads, the damage caused by the high loads corresponding to the left-hand part of the endurance curve, is usually negligibly small, and then this simpler form of the equation can be used:

$$N = \alpha(S - E)^{-\beta} \tag{35a}$$

The equation can also be written in two other ways:

$$\log (S - E) = \log b - c \log N \tag{35b}$$

and

$$S = bN^{-c} + E \tag{35c}$$

where

$$c = 1/\beta \tag{36}$$

$$b = \alpha^{1/\beta} = \alpha^{c} \text{ or } \alpha = b^{1/c} \tag{37}$$

In the form (35c) this endurance function has previously been proposed by Palmgren.[34]

It should be observed that the numerical values of the parameters b, α and E are dependent on how S is defined. In this paper S is normally defined as the relative load amplitude, s_a, or as the relative maximum load, s_{max}. The endurance limit is then denoted s_{aE} or $s_{max\ E}$, respectively.

A convenient way of obtaining the parameters b, c and E is by calculating them from S-values corresponding to three values of N, for instance $N = 5 \times 10^4$, 5×10^5 and 5×10^6. For these values the following formulae are easily derived from Eq. (35c).

$$c = \log (\Delta S_1/\Delta S_2) \tag{38}$$

$$b = f \Big/ \left(\frac{1}{\Delta S_2} - \frac{1}{\Delta S_1}\right) \tag{39}$$

$$E = S_{5 \times 10^6} - \left[1 \Big/ \left(\frac{1}{\Delta S_2} - \frac{1}{\Delta S_1}\right)\right] \tag{40}$$

where

$$\Delta S_1 = S_{5 \times 10^4} - S_{5 \times 10^5}$$
$$\Delta S_2 = S_{5 \times 10^5} - S_{5 \times 10^6}$$
$$f = \text{the antilog of } c \log (5 \times 10^5)$$

The S-values can be obtained by graphical interpolation between the test values in a usual S-log N-diagram. The reason for choosing the particular N-values used above is that the maximum damage intensity for gust loads normally occurs within the range $N = 5 \times 10^4$ to $N = 10^7$ (compare Table II). Any other three N-values, differentiated by factors of 10, and corresponding S-values can be used.

It is also possible to determine the parameters b, c and E by graphical methods, as will be evident from the diagrams of the following section.

6.4. *Different P-Cases*

Weibull has recently found[35] that in a number of applications the parameter c is independent of the probability P. When this is the case all the P-S-N-curves will become straight lines in the co-ordinate system S vs. N^{-c} according to Eq. (35c). This observation has led Weibull to define two specific cases, (A) and (B) given below, for which the different P-curves satisfy certain conditions in respect of the variation of the scatter along the S-N-curve as defined by the parameters b, c and E. A number of other P-cases can also be defined. The following simply definable cases might be of particular interest.

Case A: b and c are independent of P, while E varies. This implies that the standard deviation in S is independent of N.

Case B: c and E are independent of P, while b varies. This implies that the standard deviation in $\log (S - E)$ is independent of N and the standard deviation of $\log N$ is independent of S.

Case C: b and E are independent of P, while c varies. The coefficient of variation in $\log N$ is independent of S.

Case D: c is independent of P, the standard deviation in $\log S$, and thus the coefficient of variation in S, is independent of N. This implies that the ratio between the S-ordinates for various P-curves is constant, thus for instance

$$S_P/S_{50} = \text{constant}$$

Case E: $E = 0$ and c is independent of P. This is a special case of both B and D. The standard deviation in $\log S$ is independent of N and the standard deviation of $\log N$ is independent of S.

In order to find out which, if any, of these cases is applicable, plotting in the following co-ordinate systems can be used:

(a) $\log S$ versus $\log N$
(b) $\log (S - E_{50})$ versus $\log N$
(c) $\log (S - E_P)$ versus $\log N$
(d) S versus N^{-c}

where E_{50} is the endurance limit for the mean curve and E_P the endurance limit for the P-curve under consideration. The above sequence also indicates the order in which plotting should usually be done in order to identify any of the cases. In Fig. 15a–15d the four methods of plotting are illustrated.

The following comments concerning the cases and the illustrations might be of interest:

(*a*) *log S versus log N.*

A family of P-S-N-curves will not usually display any marked regularity in this plot. If, however, the curves are straight and parallel lines, the special case E is valid. If the curves are not straight lines, but are a constant distance apart in the ordinate direction, case D is valid. Furthermore, for any particular P-curve appearing as a straight line, $E_P = 0$.

(*b*) *log* $(S - E_{50})$ *versus log N.*

By trial and error that value of E_{50} should be determined, which with the best fit transforms the mean curve to a straight line. The extent to which this succeeds is a measure of the accuracy with which the basic function, Eq. (35) is valid for the mean curve. As the cases D and E have been investigated by plot (a), only the cases A, B and C need further investigation. Case B is valid if all the P-curves are straight and parallel. If the lines are straight and their extensions intersect each other at $\log N = 0$ case C is valid. Case A might be valid if the P-curves, except the mean curve, are not straight.

(*c*) *log* $(S - E_P)$ *versus log N.*

If all P-values fall on a single straight line, case A is valid. If the curves of the family are straight and parallel, case B or case D may be valid. If at the same time the curves in plot (a) have constant displacements from each other in the vertical direction, the validity of case D has been finally established. If the P-curves are straight but not parallel, this in the first place only

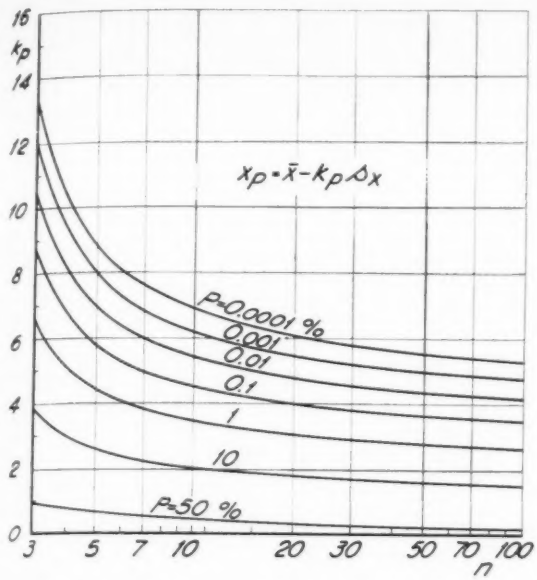


Fig. 13. Factor k_p versus number of test specimens, n, for different P-values and a confidence of 0.90.

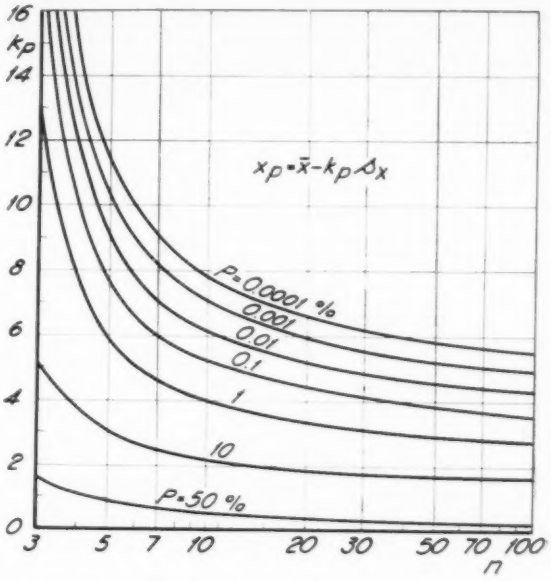


Fig. 14. Factor k_p versus number of test specimens, n, for different P-values and a confidence of 0.95.

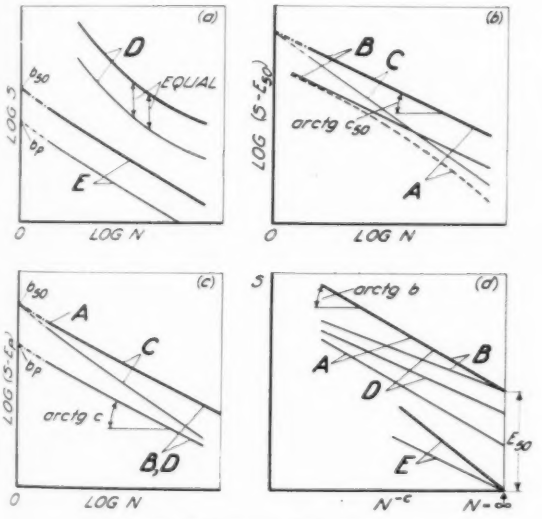


Fig. 15. Different P-Cases

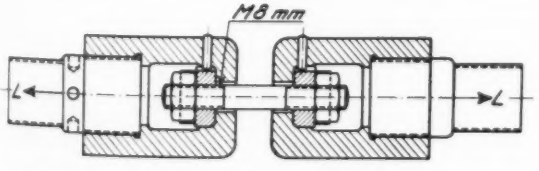


Fig. 16. Threaded bolt mounted in fixture for testing.

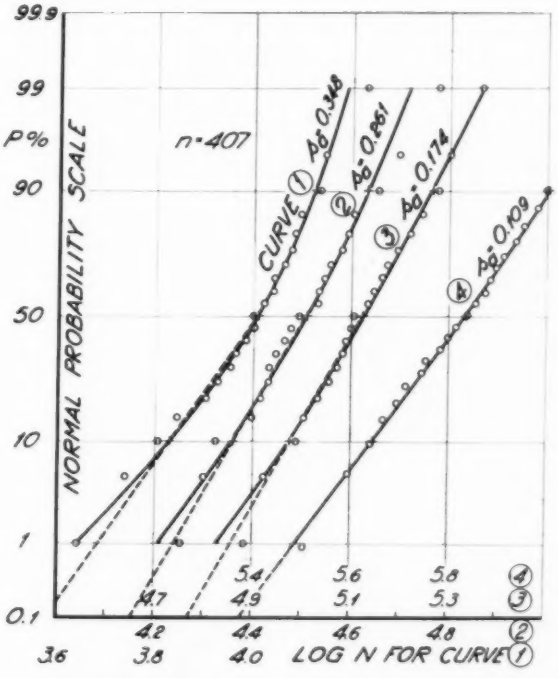


Fig. 17. Results from pulsating tensile tests on threaded bolts.

proves the validity of the basic S-N-function for all the P-curves.

(*d*) *S versus* N^{-c}.

This plot is of interest for all cases for which the parameter c is independent of P—i.e., cases A, B, D and E, for which all the P-curves will appear as straight lines. The plot does not in principle reveal anything not obtainable by the three other methods, but it might be practical to try this plot before plot (c), after the value of c_{50} has been obtained from plot (b). The plot (d) is of an appreciable interest in the first place, because it gives a further check on the validity of any particular P-case and secondly, because the diagram includes $N = \infty$ which enables direct determination of E. Case A is valid if all the P-lines are parallel and case B if they all intersect at $N = \infty$, and $S = E_{50}$.

It should be pointed out here that it is doubtful whether any of these P-cases, or any other P-cases, can in reality be valid for more than a limited part of the S-N-curves. Case A, for instance, would imply that lines for very small P-values, at least when normal distribution is assumed, would cut the abscissa at finite N-values, which of course, is absurd. Furthermore, case B would imply that there is no scatter whatsoever in the endurance limit, which also seems to be a rather questionable occurrence. These circumstances naturally limit the scope of validity for the P-cases defined above. It is believed that the concept of P-cases would still be of great importance provided, of course, that rules can be found for tying up certain types of structural elements and loadings with certain P-cases within regions of practical significance.

During the completion of this work a paper of Head[36] has come to the author's attention. For a series of test results, Head has studied four P-cases, of which three can be defined as the above cases A, B and D, in a numerical way without plotting the curves. There is, however, usually a need for plotting the P-curves whenever one wishes (1) to check the validity of the above used (or any other) S-N-function, and (2) to obtain graphically the parameters b, c and E. As indicated in Section 6.3, these parameters can also be obtained from any three values of the S-N-function, but if more than three values have been determined by tests, a graphical determination of at least the mean curve is preferable.

6.5. *Results of Tests with Large Sample Sizes*

In order to check accurately the validity of any P-case and to determine the parameters b, c and E it is necessary to have rather large sample sizes for a not too small number of load levels. It is therefore desirable to carry out S-N-testing on structural elements, which are typical for airplane designs, in order to find out whether or not any general rules or trends can be found regarding the type of loading and the material and detail design of the element on one hand and the type of P-case on the other. Once such general rules or trends are found, it will be possible to limit the required sample size.

Very few test results with sample sizes of a magnitude required for checking P-cases are available to-day. Only three series of tests with fairly large sample sizes have been available to the FFA. The variation as regards the applicability of P-cases will be briefly described in the following.

1. *Pulsating tension tests of threaded bolts.*

These tests, which have been run at the Royal Institute of Technology, Stockholm, consisted mainly of 49 static tests and 407 fatigue tests at four different load levels. The test specimen is shown in Fig. 16. In all the fatigue tests the mean load was constant and equal to 2,205 lbs. The test results, which have been treated statistically by Weibull,[37] are plotted on a normal probability paper (Fig. 17). The values of log N for $P = 50$ and 10 per cent have been plotted in Figs. 18 and 19. In these figures the relative load amplitude has been used, obtained by division by the mean value of the static ultimate strength, 4,670 lbs.

From Fig. 18 it can be seen that the points for 50 and 10 per cent probability fit rather closely to straight parallel lines with the exception of the points at the highest load level, where a small deviation is observed. This deviation, however, can be explained by the fact that these two points fall above the point of inflection of the endurance curve (see Section 6.3). The said two P-lines being parallel, indicates the probability case E.

In Figs. 18 and 19 probability lines have also been calculated for P-values lower than those directly obtainable from Fig. 17. This has been done by assuming a log-normal distribution, which means that the somewhat curved lines fitting the test points in Fig. 17 have been approximated by straight lines as indicated in this figure. In this approximation only the test results below the mean values have been taken into account.

2. *Pulsating tension tests of 75S-T notched specimens.*

The specimens were of 10 mm. diameter circular cross section and the notch consisted of a 3-mm. hole. The investigation, which has been carried out at the FFA, consisted of 10 static tests and 387 fatigue tests at seven load levels. The mean value of the ultimate static strength was 94.16 kips per sq.in. For all the fatigue tests the lower load limit was zero. The results have been treated statistically in reference 35, from which Figs. 20 and 21 have been derived.

The figures show that the test results fit the P-case A with fairly good accuracy. Also in these two figures probability curves for rather low P-values have been calculated, but as the sample size at each load level was comparatively small, the lower P-curves have been obtained by the method of pooling to be described in the next section.

3. *Reversed torsion of copper wires.*

This test series has been carried out by Ravilly[38] and Weibull has dealt with the results statistically.[35] The investigation comprised 20 tests at each of ten stress levels. The method of testing implied a constant strain loading instead of the method of constant stress loading commonly used.

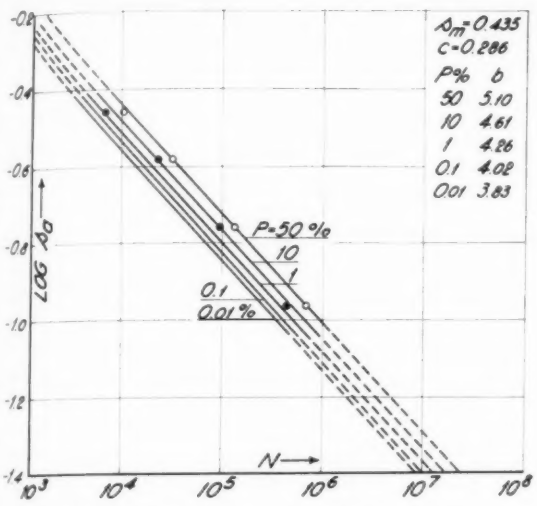


Fig. 18. P-log Δ_σ-N diagram for threaded bolts, loaded by pulsating tension. Points shown are obtained from Fig. 17. P-Case E.

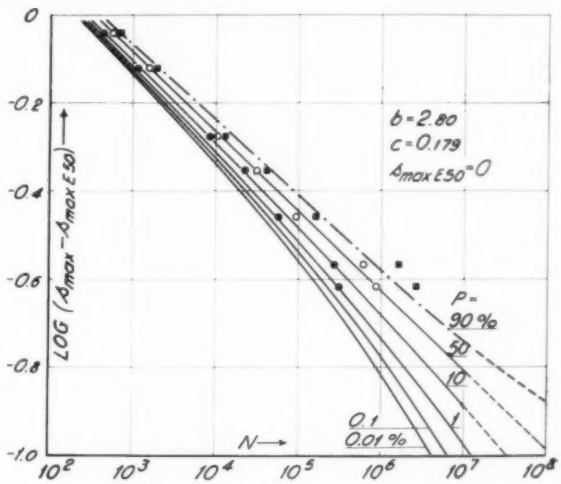


Fig. 20. P-log (Δ_{max}-$\Delta_{max\,E50}$)-N diagram for 75S-T notched specimens, loaded by pulsating tension. Points shown are obtained from distribution curves. P-Case A.

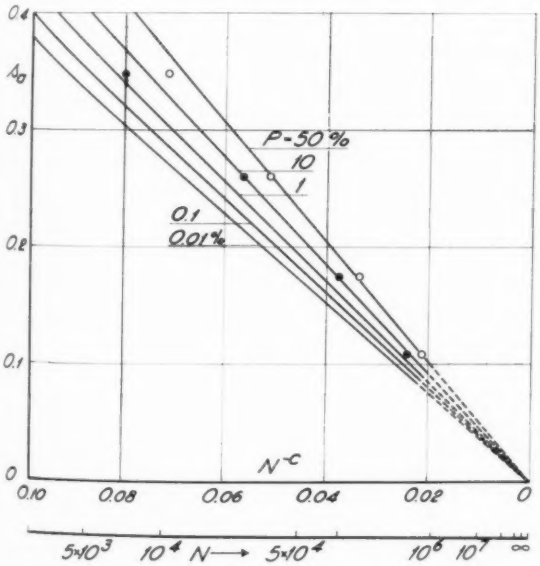


Fig. 19. P-Δ_σ-N^{-c} diagram for threaded bolts. P-Case E.

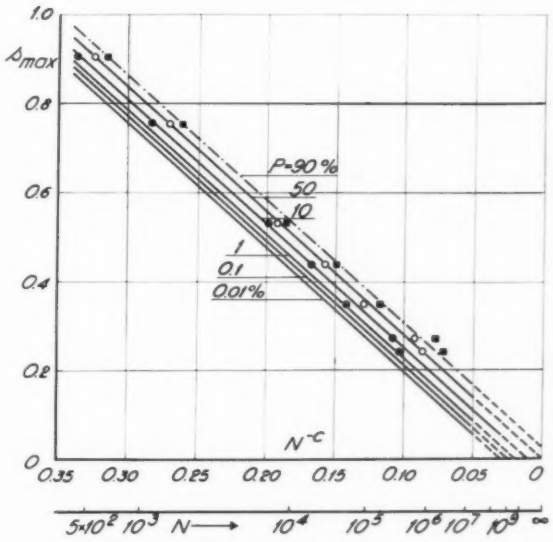


Fig. 21. P-Δ_{max}-N^{-c} diagram for 75S-T notched specimens. P-Case A.

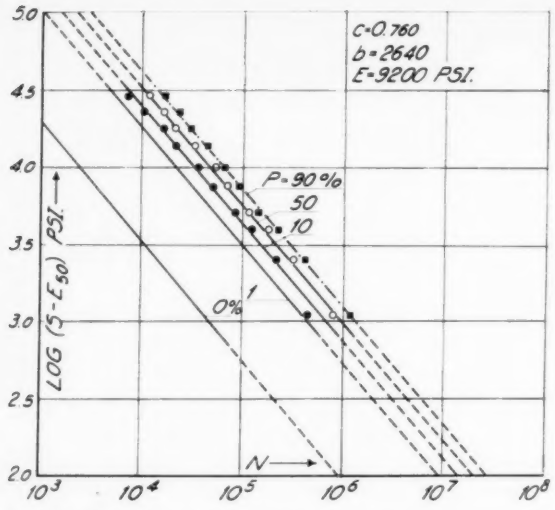


Fig. 22. P-log (S-E_{50})-N diagram for copper wires loaded by reversed torsion. Points shown are obtained from distribution curves. P-Case B.

The results are plotted in Figs. 22 and 23, which indicate that the P-case B is applicable with rather good agreement. Also for these test results the method of pooling has been applied, but in this instance Weibull's distribution function has been used, which explains that the curve for $P = 0$ is visible in Fig. 22 and is an inclined line in Fig. 23.

6.6. *Pooling Test Results*

Most S-N-testing hitherto performed at laboratories throughout the world has been carried out with sample sizes at each load level, which are much too small to yield determination of the mean value and the standard deviation in log N with any accuracy. If, however, for a particular test series a certain P-case can be assumed to be valid, Weibull has shown[35] that in principle all test values at all load levels can be utilized in order to support the determination of the various statistical parameters. This method of pooling the test results will be briefly explained and exemplified.

After all the test values, and the mean values for each load level, have been plotted in a S vs. log N diagram, a curve is drawn so that it fits to the mean values as closely as possible. This curve can be said to represent the $P = 50$ per cent curve with a confidence of 0.5. The parameters b, c and E are determined for this curve, for instance by using the Eqs. (38) to (40). Then, for the pooling procedure, all test values are excluded which are either suspected to lie above the point of inflection of the S-N curves or represent specimens that have not failed. If, now, there is reason to assume that a certain P-case is valid, those deviations from the mean curve are calculated which, according to the definition of the P-case, should be independent of N. For instance, if case B or E is valid, the deviations in log N are computed and from all these deviations the standard deviation in log N is obtained, whereupon the different P-curves are drawn in the diagram.

If case A or D is assumed to be valid, the deviation from the mean curve of each test value in S or log S, respectively, is calculated. These deviations yield the standard deviation in S and log S, whereupon the P-curves are obtained. The procedure is somewhat more complicated in case C, but the pooling of test values from different stress levels can, in principle, be carried out in the same manner as has been briefly described for the other P-cases.

As pointed out, this method of pooling presupposes that a certain P-case can be assumed to be valid. If the sample size is small, it is not possible to ascertain a particular P-case only by means of the test values. There is, however, a possibility that extended research could lead to the establishment of some general rules which tie up certain P-cases with certain types of loadings and structural elements, whereby the stress concentration factor might play an indicative role. At present the available test results are insufficient to enable any concrete conclusions regarding relationships between P-cases and types of elements and loadings to be drawn.

Apart from this possibly useful application, the pooling method has a definite advantageous use in order to support the determination of low P-levels, which implies an improved accuracy in the determination of the standard deviation, provided that the sample size is sufficiently large for determining a certain P-case. This has been exemplified by the test series with threaded bolts described in Section 6.5. As the results of these tests have very clearly indicated that case E is valid, pooling of the test values has been made and

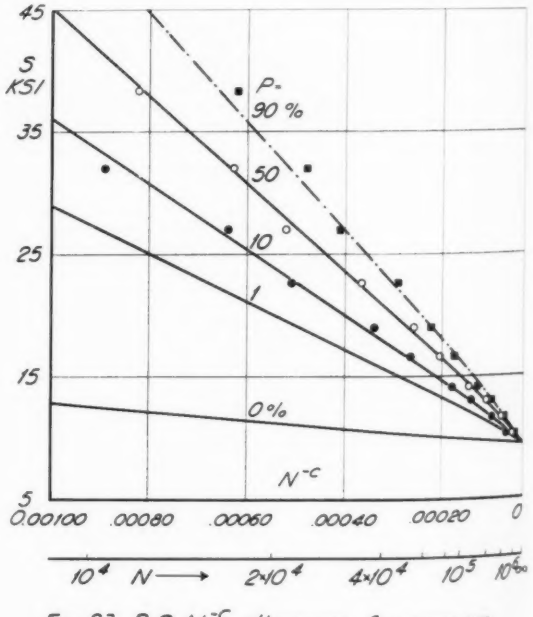


Fig. 23. P-S-N^{-c} diagram for copper wires. P-Case B.

the result is shown in Fig. 24. On this distribution curve a quite low probability value, $P = 0.32$ per cent, is directly plotted and the standard deviation may thus be computed with a fair accuracy, enabling the determination in particular of the curves for $P = 1$ and 0.1 per cent in Figs. 18 and 19 with an improved degree of certainty.

6.7. *Studies of the Parameters in the P-S-N-Function*

It follows from Eq. (35) that the S-N curve is defined by three parameters (either b, c and E or α, β and E), provided that only the part of the S-N curve to the right of the point of inflection is considered. In Table I these parameters valid for s_a-N or s_{max}-N curves, have been listed for a number of S-N tests carried out in Sweden and elsewhere.[39–43] The test results selected have been limited to tests with particular interest for airplane design—i.e., tests which are run with either constant mean loads or with the lower load limit equal to zero. For the first mentioned group the table discloses a mostly rather consistent trend in the variation of the five parameters with the mean load. It might be pointed out that, with the exception of test series numbers 15 and 17 in the table, the sample sizes have been much too low to compute the parameters with a high degree of accuracy.

The table also shows the standard deviation for those tests for which the sample size was not too small to allow pooling of the results according to the previous section. For this pooling the P-case D has been applied, which involves a rather arbitrary assumption as the sample size at each load level was too small to prove any special P-case with any certainty. For this reason the tabled values of the standard deviation are, of course, also uncertain, but with this reservation the values indicate larger standard deviation for the more complicated built-up elements, such as riveted joints, than for the simpler ones with simple notches, such as holes or cut-out notches.

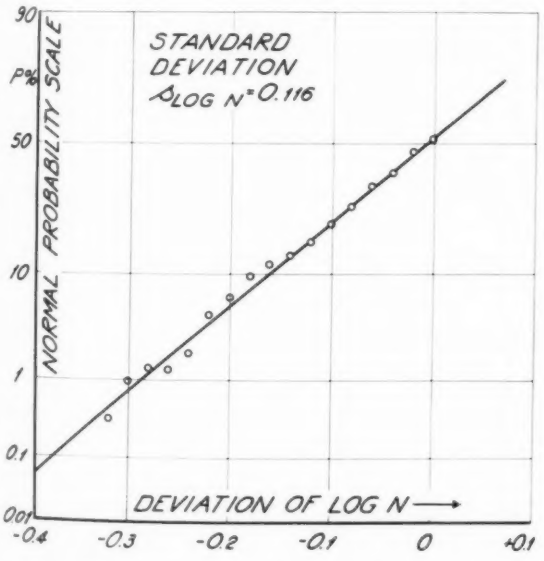


Fig. 24. Deviations of log N from median s_a-N curve for threaded bolts. Class interval 0.02. 308 test specimens.

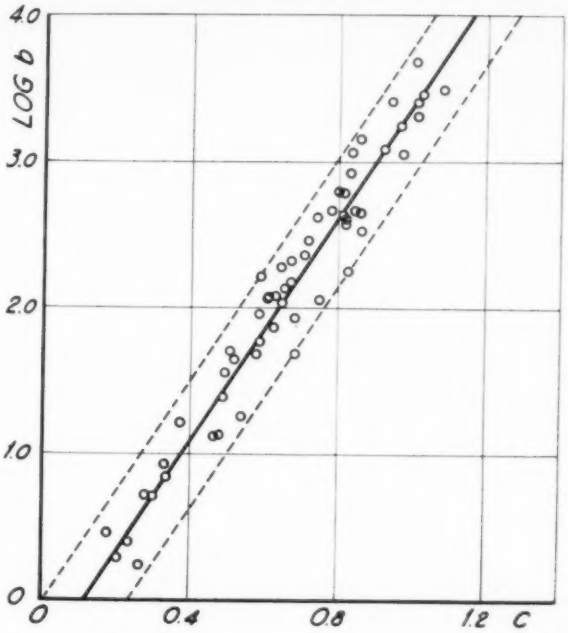


Fig. 25. Log b vs. c for specimens listed in Table I.

In order to determine whether any conformity applies for the various parameters it might be of interest to plot the values in different ways. Fig. 25 shows a plot of log b over c and Fig. 26 a plot of log α over β. Fig. 25 discloses a rather consistent correlation between the plotted quantities, as all the points fall rather close to a straight line within a fairly narrow band. The equations for the mean line and the upper and lower boundaries in Fig. 25 can all be written in the form

$$\log b = 3.88\, c - \log b_0 \qquad (41)$$

where $\log b_0 = 0.463$ for the mean curve, 0.056 for the upper line and 0.948 for the lower line.

Using the formula (35c) in Chapter 6.3 for the relative load amplitudes (s_a and s_{aE}) and eliminating the parameter b, we obtain the following equations:

For the mean line, $s_a = 0.344\ (N\ 7{,}600)^{-c} + s_{aE}$.
For the upper line, $s_a = 0.879\ (N\ 7{,}600)^{-c} + s_{aE}$.
For the lower line, $s_a = 0.113\ (N\ 7{,}600)^{-c} + s_{aE}$.

Each of these equations represents a family of curves with a certain intersection point, which, for example, for the mean line is $s_a - s_{aE} = 0.344$ and $N = 7{,}600$.

In Fig. 26 the corresponding values of the parameters, log α and β, are plotted. The mean and boundary lines in Fig. 25 correspond to straight lines also in Fig. 26, as shown by the figure. This plot indicates that $\alpha = 1{,}000$ is a rather representative overall figure for this parameter.

TABLE 1.
Details of specimens and S-N-parameters determined by tests

No.	Type of specimen	Stress concentration factor K_t [1]	Material	Reference	Average tensile ultimate strength of the specimen ksi [2]	Mean load s_m	b	S-N-parameters c	α	β	s_{aE}	S-N curve determined from: number of specimens	S-N curve determined from: number of load-levels	Pooling-procedure based on: number of specimens	Pooling-procedure based on: number of load-levels	Standard deviation $s_{\log s_a}$
1	Strip with two edge-cut notches	2.0	24S-T3	NACA TN 2389 (Ref. 39)	74.5	0	149	.669	1,769	1.495	.165	11	7			
						.134	838	.828	3,389	1.208	.129	12	9			
						.268	2,550	.940	4,233	1.064	.124	13	9			
						.403	4,770	1.005	455	.995	.121	10	7			
2	Strip with two edge-cut notches	4.0	24S-T3	NACA TN 2389 (Ref. 39)	65.4	0	74	.622	1,006	1.608	.089	8	8			
						.153	85	.679	688	1.472	.076	13	8			
						.306	112	.748	548	1.337	.073	6	6			
						.459	327	.860	837	1.162	.065	7	7			
3	Strip with two edge-cut notches	5.0	24S-T3	NACA TN 2390 (Ref. 40)	62.4	0	24	.488	680	2.050	.080	12	11			
						.160	435	.809	1,826	1.236	.063	10	10			
						.320	1,210	.921	2,220	1.086	.048	8	8			
						.400	1,750	.964	2,311	1.038	.047	7	7			
						.481	2,550	1.011	2,348	.989	.047	9	9			
4	Strip with two drilled holes	2.2	24S-T Alclad sheet	FFA Report 18 (Ref. 41)	59.7	0	51	.509	2,239	1.965	.100	15	5			
						.214	210	.670	2,914	1.492	.075	15	5	8	3	.035
						.321	405	.740	3,327	1.351	.076	16	5			
						.538	136	.653	1,842	1.531	.072	15	5			
5	Specimen of circular cross-section notched by a drilled hole	2.05	24S-T Extruded	FFA Report 48 (Ref. 43)	68.7	-.103	2.0	.204	26.4	4.902	.153	14	6			
						0	2.5	.236	48.5	4.237	.139	15	7			
						.228	5.1	.301	224	3.322	.055	14	5	6	2	.030
						.559	7.0	.332	351	3.012	.013	11	4			
6	Double shear riveted joint with loosely driven rivets	-	24S-T Alclad sheet	FFA Report 28 (Ref. 42)	61.2	0	163	.588	5,827	1.702	.082	24	6			
						.151	187	.641	3,492	1.560	.018	31	7	11	4	.121
						.302	48	.577	808	1.733	.019	21	7	9	4	.063
						.512	13.4	.477	229	2.096	.014	16	5			
7	Strip with two edge-cut notches	2.0	75S-T6	NACA TN 2389 (Ref. 39)	87.5	0	458	.778	2,622	1.285	.159	13	8			
						.114	608	.813	2,662	1.230	.132	9	8			
						.229	458	.841	1,457	1.189	.104	10	10			
						.343	447	.857	1,233	1.167	.088	10	10			
8	Strip with two edge-cut notches	4.0	75S-T6	NACA TN 2389 (Ref. 39)	82.5	0	13.2	.462	265	2.163	.075	9	8			
						.121	377	.818	1,416	1.223	.053	10	7			
						.242	1,133	.970	1,409	1.031	.029	8	6			
						.363	3,083	1.079	1,710	.927	.028	6	5			

1) K_t is based upon the net area.

2) Unless otherwise stated, this stress is based upon the net area.

TABLE 1. - Continued.

No.	Type of specimen	Stress concentration factor K_t [1]	Material	Reference	Average tensile ultimate strength of the specimen ksi [2]	Mean load s_m	b	S-N-parameters c	α	β	s_{aE}	S-N curve determined from: number of specimens	S-N curve determined from: number of load-levels	Pooling-procedure based on: number of specimens	Pooling-procedure based on: number of load-levels	Standard deviation $s_{\log s_a}$
9	Strip with two edge-cut notches	5.0	75S-T6	NACA TN 2390 (Ref. 40)	77.5	0	1.8	.265	8.6	3.774	.045	16	16			
						.129	18.2	.539	218	1.857	.037	11	10			
						.258	48.2	.681	295	1.468	.034	9	9			
						.387	175	.824	528	1.214	.033	8	8			
10	Riveted butt joint	-	75S-T Sheet	FFA Report 28 (Ref. 42)	53.1 based on gross area	0	8.4	.330	635	3.034	.060	23	9			
						.134	35.8	.492	1,435	2.032	.052	18	7	11	3	.062
						.268	58.6	.586	1,047	1.708	.038	27	8	11	4	.078
11	Strip with two edge-cut notches	2.0	SAE 4130 Steel	NACA TN 2389 (Ref. 39)	117.0	0	89	.581	2,266	1.720	.219	10	7			
						.085	116	.606	2,582	1.652	.214	14	8			
						.171	118	.609	2,521	1.641	.208	11	9			
						.256	118	.609	2,521	1.641	.208	9	7			
12	Strip with two edge-cut notches	4.0	SAE 4130 Steel	NACA TN 2389 (Ref. 39)	129.0	0	120	.628	2,024	1.591	.107	11	10			
						.077	623	.797	3,203	1.255	.104	10	8			
						.155	2,020	1.011	1,861	.989	.103	11	8			
						.233	31 100	1.204	5,381	.831	.100	8	8			
13	Strip with two edge-cut notches	5.0	SAE 4130 Steel	NACA TN 2390 (Ref. 40)	121.8	0	44	.518	1,464	1.932	.083	12	11			
						.082	107	.642	1,438	1.557	.082	11	11			
						.164	406	.813	1,617	1.230	.082	9	9			
						.246	2,920	1.021	2,474	.979	.081	10	10			
14	Strip notched by a a central, drilled hole	2.3	Cr-Mo-steel	FFA Report 28 (Ref. 42)	151.0	0	230	.704	2,272	1.421	.211	21	8			
						.094	286	.717	2,675	1.395	.156	16	9	9	6	.021
						.283	1,416	.857	4,771	1.167	.114	18	6	8	4	.042
						.472	1,151	.831	4,821	1.203	.082	19	8			
15	Threaded bolt, see Fig. 16	-	Steel	(Ref. 37)	4670 lbs	.435	5.1	.286	298	3.497	0	407	4	308	3	.116 [3]
16	Riveted butt joint	-	75S-T Sheet	FFA Report 28 (Ref. 42)	53.1 based on gross area	s_{min} = 0 [4]	15.9	.387	1,272	2.584	s_{maxE} .140	23	10	17	7	.039 [5]
17	Specimen of circular cross-section notched by a drilled hole	2.0	75S-T Extruded	SAAB TN 30 (Ref. 35)	94.2	s_{min} = 0 [4]	2.80	.179	315	5.587	s_{maxE} 0	387	7	270	5	.0213 [5]

1) K_t is based upon the net area

2) Unless otherwise stated, this stress is based upon the net area.

3) The standard deviation is computed in log N.

4) The lower load limit is equal to zero.

5) The standard deviation is computed in s_{max}.

7. Cumulative Damage and Life Calculations

As early as in 1924 Palmgren, in connection with the design of ball bearings, was confronted with the problem of calculating the fatigue life of an element subjected to varying stress amplitudes. His work led him to the proposal of the following, now so well-known, formula for failure due to the action of accumulated fatigue.[34]

$$\Sigma(n_i/N_i) = 1$$

where

n_i = number of applied stress cycles of a certain amplitude

N_i = number of stress cycles for failure at the same amplitude

This equation implies that failure occurs when the sum of a number of "part damages" is equal to one, thus

$$D_{\text{failure}} = \Sigma D_i = \Sigma(n_i/N_i) = 1 \qquad (42)$$

It has been proved experimentally that this equation does not hold for an arbitrary sequence of stress amplitudes.[42, 44–49] There is some indication, however, that the cumulative damage theory might turn out to be a relatively good approximation for the accumulated action of repeated loads with varying amplitudes, provided that the higher and lower loads are well mingled with each other, or, in other words, that the loading in effect is equivalent to the case where a limited "program unit" is repeated a great number of times.*

Eq. (42) can be written in the form

$$D = \Sigma[(-\Delta H)/N] \qquad (43)$$

where $(-\Delta H)$ is the number of gust load cycles within a certain interval, Δs_a (Fig. 27), and N the corresponding life of the element.

For an infinitesimal increment in s_a, Eq. (43) is transformed into

$$D = -\int \frac{dH}{ds_a}\frac{1}{N}\,ds_a \qquad (44)$$

Introducing Eq. (23b) of Section 5.2 and Eq. (35a) of Section 6.3, the following approximate formula has been deduced in Appendix A by S. Eggwertz at the FFA:

$$D = \frac{H_0}{\alpha} h^{-\beta} e^{-hs_{aE}} \Gamma(\beta + 1) \qquad (45)$$

where H_0 and h are parameters of the gust load curve (see 5.2), while α, β, and s_{aE} refer to the endurance curve (see 6.3). The Gamma-function $\Gamma(\beta+1)$ can be obtained from mathematical tables.[50–52]

The error introduced by the above approximate formula, in comparison with the more accurate solution obtained in Appendix A, depends upon the numerical values of the five said parameters. In most cases the error seems to be only a few per cent. As gust load curves and endurance curves are never known with great accuracy, Eq. (45) may be used for all practical purposes.

* This problem will be treated by W. Weibull in a FFA report to be published.

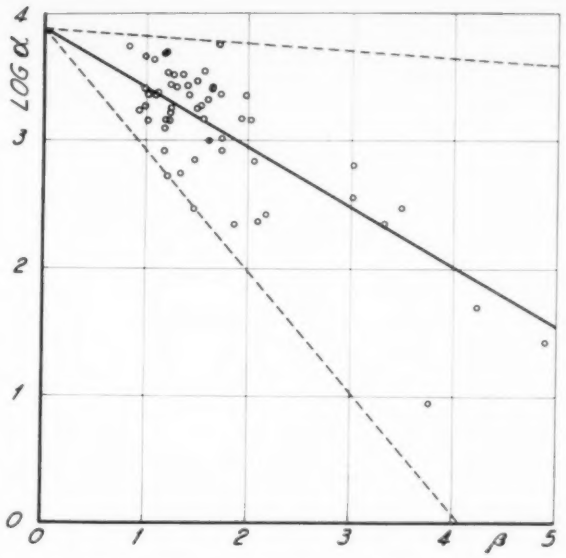


Fig. 26. Log α vs. β for specimens listed in Table I.

If it is assumed that the above value of H_0 corresponds to a flown distance of 10^6 miles (Section 5.2), the life, L, of the structure can be obtained from

$$L = 10^6/D \text{ miles}$$

or, introducing Eq. (45)

$$L = \frac{10^6}{H_0\Gamma(\beta + 1)}\,\alpha h^{\beta} e^{hs_{aE}} \qquad (46)$$

It is rather important to know at which range of stress amplitudes most of the fatigue damage is created, as the fatigue testing should, of course, be mainly con-

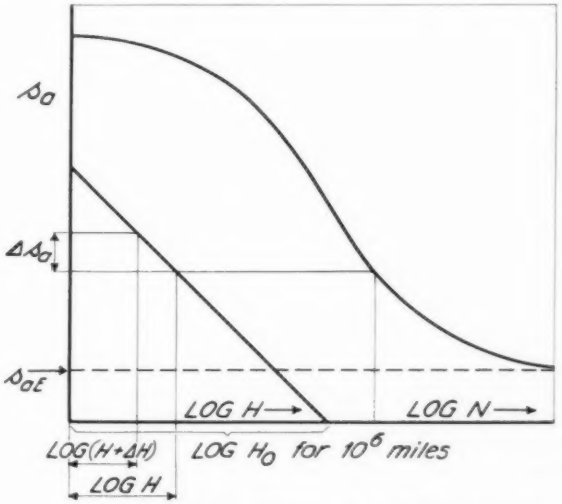


Fig. 27. Definitions for Chapter 7.

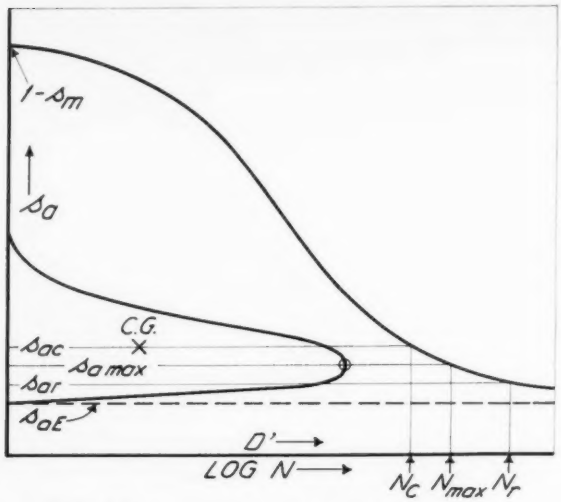


Fig. 28. Damage intensity curve.

centrated to this range. The damage intensity may be written (Appendix A)

$$D' = -\frac{dH}{ds_a}\frac{1}{N} = \frac{H_0 h}{\alpha} e^{-hs_a}(s_a - s_{aE})^{\beta} \qquad (47)$$

Fig. 28 shows a typical form of a damage intensity curve. Calculations have indicated that for values of s_a, due to gusts, which correspond to $N < 10^4$ the damage contribution is in most cases negligible. Some s_a levels related to the damage intensity function in a specific way might be of special interest in order to find out whether there exists one significant s_a-level, such that the results of testing at this single level would be representative for the cumulative damage due to loading according to the whole load spectrum.

One s_a level of interest is represented by the maximum of the function.

$$s_{a\,max} = \beta/h + s_{aE} \qquad (48)$$

The corresponding value of N is

$$N_{max} = \alpha(h/\beta)^{\beta} \qquad (49)$$

As illustrated in Fig. 28 the damage intensity maximum is usually rather close to the endurance limit.

Another s_a level of possible interest is the centroid of the area enclosed by the damage intensity curve

$$s_{ac} = (\beta + 1)/h + s_{aE} \qquad (50)$$

$$N_c = \alpha[h/(\beta + 1)]^{\beta} \qquad (51)$$

In Table II values of L, $s_{a\,max}$, N_{max}, s_{ac} and N_c have been calculated for a number of structural elements using the mean endurance curves and the basic gust curve defined in Chapter 5.

Finally, it may be of some interest to define a reduced stress amplitude, s_{ar}, as the constant amplitude which would require H_0 (compare Section 5.2) stress cycles to yield the same damage as the entire gust load spectrum, containing H_0 total number of gust cycles. The corresponding value of N is obtained directly from this definition

$$N_r/H_0 = \frac{1}{D} \qquad (52)$$

Eq. (45) gives

$$N_r = \frac{\alpha h^{\beta} e^{hs_{aE}}}{\Gamma(\beta + 1)} \qquad (53)$$

By Eq. (35c) of Section 6.3, the corresponding "reduced" stress amplitude is obtained

$$s_{ar} = \frac{[\Gamma(\beta + 1)]^c}{h} e^{-hcs_{aE}} + s_{aE} \qquad (54)$$

where

$$c = 1/\beta$$

It can be shown that this stress level normally falls much below $s_{a\,max}$ and usually immediate above s_{aE}. It therefore seems to be an open question whether this reduced stress amplitude is of a significant practical value. Anyhow, the stress level is too low to be used for representative constant amplitude testing.

In order to calculate the fatigue life by using the common cumulative damage theory (or any other damage theory) it seems reasonable to use a P-S-N-function with the same P-value as the element probability aimed at. The question whether this method is correct and, if not, which other assumption should be made, ought

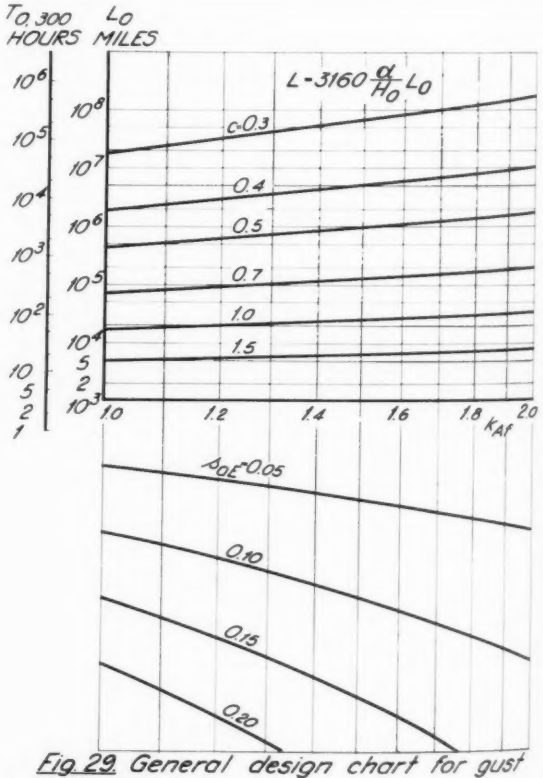


Fig. 29. General design chart for gust fatigue life.

TABLE II.

Summary of calculated values for specimens according to table I.

Calculations carried out for P = 50%, a confidence of 0.50 and under the assumption that $n_D = 3.75$ and $\nu,\omega,\ v = 1.0$.

Type of specimen -defined in table I	Fatigue factor k_A	S-N-parameters b	c	α'	β	σ_{aE}	L in miles	$L/L_{k_A=1}$	$\sigma_{a\,max}$	N_{max}	σ_{ac}	N_c	$\sigma_{a\ 2\times10^6}$	$d\sigma_a$
1	1.0	2,540	.940	4,188	1.064	.124	4.72×10^{7}	1.0	.144	2.61×10^{5}	.164	1.29×10^{5}	.127	.008
	1.2	1,970	.911	4,134	1.098	.125	2.83×10^{8}	6.0	.143	3.47×10^{5}	.159	1.79×10^{5}	.129	.009
	1.4	1,570	.885	4,083	1.130	.126	1.45×10^{9}	30.7	.142	4.50×10^{5}	.155	2.20×10^{5}	.130	.011
	1.6	1,280	.862	4,024	1.160	.127	7.44×10^{9}	158	.141	5.69×10^{5}	.153	2.77×10^{5}	.132	.011
2	1.0	101	.726	576.4	1.377	.073	1.53×10^{6}	1.0	.100	8.52×10^{4}	.119	4.01×10^{4}	.076	.004
	1.2	93	.707	608.4	1.414	.075	5.41×10^{6}	3.54	.098	1.26×10^{5}	.114	6.01×10^{4}	.078	.006
	1.4	88	.694	633.6	1.441	.075	1.72×10^{7}	11.2	.095	1.79×10^{5}	.109	8.40×10^{4}	.079	.006
	1.6	83	.685	633.5	1.460	.076	5.38×10^{7}	35.2	.094	2.30×10^{5}	.106	1.08×10^{5}	.080	.006
3	1.0	945	.895	2,111	1.117	.050	7.12×10^{5}	1.0	.072	1.53×10^{5}	.091	7.51×10^{4}	.052	.006
	1.2	725	.864	2,044	1.157	.055	2.25×10^{6}	3.16	.074	2.06×10^{5}	.090	9.99×10^{4}	.058	.006
	1.4	580	.839	1,967	1.192	.059	6.61×10^{6}	9.28	.075	2.63×10^{5}	.089	1.27×10^{5}	.062	.006
	1.6	460	.815	1,851	1.227	.062	2.08×10^{7}	29.2	.077	3.25×10^{5}	.089	1.57×10^{5}	.065	.008
4	1.0	302	.709	3,146	1.410	.075	1.01×10^{7}	1.0	.102	5.07×10^{5}	.122	2.38×10^{5}	.085	.018
	1.2	220	.676	2,918	1.479	.075	3.39×10^{8}	3.36	.099	7.36×10^{5}	.115	3.42×10^{5}	.087	.019
	1.4	180	.655	2,775	1.527	.077	1.22×10^{8}	12.1	.098	1.01×10^{6}	.112	4.67×10^{5}	.090	.020
	1.6	152	.637	2,662	1.570	.077	3.69×10^{8}	36.5	.096	1.35×10^{6}	.108	6.22×10^{5}	.092	.021
5	1.0	5.4	.309	234.6	3.236	.049	4.15×10^{7}	1.0	.112	1.85×10^{6}	.131	7.73×10^{5}	.110	.033
	1.2	5.1	.300	228.4	3.333	.056	2.40×10^{8}	5.78	.110	3.93×10^{6}	.126	1.63×10^{6}	.122	.034
	1.4	4.8	.292	207.7	3.425	.061	1.21×10^{9}	29.1	.108	7.21×10^{6}	.122	3.00×10^{6}	.130	.034
	1.6	4.5	.286	192.3	3.497	.065	6.02×10^{9}	145	.107	1.23×10^{7}	.119	5.12×10^{6}	.136	.035
6	1.0	83.1	.610	1,402	1.639	.016	4.39×10^{5}	1.0	.047	4.03×10^{5}	.067	1.84×10^{5}	.027	.016
	1.2	117	.624	2,063	1.603	.016	8.80×10^{5}	2.01	.042	7.26×10^{5}	.058	3.34×10^{5}	.030	.019
	1.4	141	.630	2,579	1.587	.017	1.75×10^{6}	3.99	.038	1.11×10^{6}	.053	5.12×10^{5}	.032	.021
	1.6	158	.632	3,013	1.582	.017	2.95×10^{6}	6.72	.036	1.58×10^{6}	.048	7.28×10^{5}	.034	.023
7	1.0	456	.848	1,367	1.179	.097	6.41×10^{6}	1.0	.120	1.18×10^{5}	.139	5.73×10^{4}	.099	.005
	1.2	465	.839	1,511	1.192	.106	4.28×10^{7}	6.68	.125	1.68×10^{5}	.141	8.14×10^{4}	.108	.006
	1.4	496	.831	1,753	1.203	.113	3.09×10^{8}	48.2	.130	2.43×10^{5}	.143	1.17×10^{5}	.116	.006
	1.6	527	.825	1,991	1.212	.120	2.47×10^{9}	385	.135	3.34×10^{5}	.147	1.61×10^{5}	.123	.008

TABLE II.- Continued

Type of specimen -defined in table I	Fatigue factor k_A	S-N-parameters b	c	α'	β	σ_{aE}	L in miles	$L/L_{k_A=1}$	$\sigma_{a\,max}$	N_{max}	σ_{ac}	N_c	$\sigma_{a\ 2\times10^6}$	$d\sigma_a$
8	1.0	1,350	.995	1,400	1.005	.028	1.06×10^{5}	1.0	.048	7.36×10^{4}	.067	3.68×10^{4}	.029	.002
	1.2	990	.950	1,423	1.053	.032	2.42×10^{5}	2.28	.049	1.04×10^{5}	.065	5.16×10^{4}	.033	.003
	1.4	817	.915	1,523	1.093	.036	6.60×10^{5}	6.23	.051	1.49×10^{5}	.065	7.34×10^{4}	.037	.004
	1.6	677	.885	1,579	1.130	.041	2.08×10^{6}	19.6	.055	2.02×10^{5}	.067	9.89×10^{4}	.043	.004
9	1.0	51.5	.692	207.7	1.445	.034	1.33×10^{5}	1.0	.062	5.25×10^{4}	.081	2.45×10^{4}	.036	.004
	1.2	37.0	.644	272.4	1.553	.034	3.09×10^{5}	2.32	.059	8.38×10^{4}	.075	3.87×10^{4}	.037	.005
	1.4	30.0	.609	266.3	1.642	.035	8.13×10^{5}	6.11	.058	1.34×10^{5}	.071	6.13×10^{4}	.039	.006
	1.6	25.0	.582	251.4	1.718	.035	1.86×10^{6}	14.0	.056	1.96×10^{5}	.068	8.91×10^{4}	.040	.007
10	1.0	59.5	.586	1,068	1.707	.038	1.35×10^{6}	1.0	.071	3.62×10^{5}	.090	1.65×10^{5}	.050	.015
	1.2	55.4	.561	1,281	1.783	.043	5.55×10^{7}	4.11	.072	7.19×10^{5}	.088	3.26×10^{5}	.059	.019
	1.4	49.7	.540	1,382	1.852	.046	1.94×10^{7}	14.4	.072	1.23×10^{6}	.085	5.54×10^{5}	.066	.022
	1.6	43.6	.520	1,421	1.923	.049	6.71×10^{7}	49.7	.072	1.98×10^{6}	.084	8.82×10^{5}	.072	.024
11	1.0	119	.610	2,526	1.640	.208	1.74×10^{10}	1.0	.240	7.28×10^{5}	.259	3.33×10^{5}	.225	.023
	1.2	119	.610	2,522	1.640	.208	2.28×10^{11}	13.1	.234	9.80×10^{5}	.250	4.48×10^{5}	.225	.023
	1.4	118	.609	2,525	1.641	.208	4.64×10^{12}	267	.231	1.27×10^{6}	.244	5.80×10^{5}	.225	.023
	1.6	118	.609	2,527	1.642	.209	∞	∞	.229	1.58×10^{6}	.241	7.24×10^{5}	.226	.023
12	1.0	69,000	1.290	5,637	.7752	.100	7.32×10^{6}	1.0	.115	1.47×10^{5}	.134	7.70×10^{4}	.101	.002
	1.2	26,000	1.180	5,517	.8475	.100	3.08×10^{7}	4.21	.114	2.10×10^{5}	.130	1.09×10^{5}	.101	.004
	1.4	11,000	1.095	4,905	.9132	.101	1.38×10^{8}	18.9	.114	2.67×10^{5}	.127	1.36×10^{5}	.102	.005
	1.6	6,000	1.031	4,619	.9699	.102	5.08×10^{8}	69.4	.114	3.45×10^{5}	.126	1.34×10^{5}	.103	.007
13	1.0	5,100	1.074	2,832	.9311	.080	2.27×10^{6}	1.0	.098	1.19×10^{5}	.117	6.06×10^{4}	.081	.003
	1.2	1,540	.958	2,124	1.044	.081	7.20×10^{6}	3.17	.098	1.51×10^{5}	.114	7.50×10^{4}	.082	.004
	1.4	740	.881	1,806	1.135	.081	2.60×10^{7}	11.5	.097	2.02×10^{5}	.110	9.88×10^{4}	.083	.005
	1.6	430	.822	1,599	1.217	.081	8.40×10^{7}	37.0	.096	2.72×10^{5}	.108	1.31×10^{5}	.084	.006
14	1.0	1,320	.850	4,692	1.177	.117	6.33×10^{7}	1.0	.140	4.03×10^{5}	.159	1.96×10^{5}	.123	.013
	1.2	1,040	.830	4,314	1.205	.125	4.48×10^{8}	7.08	.144	4.99×10^{5}	.160	2.41×10^{5}	.131	.013
	1.4	850	.806	4,310	1.241	.130	3.04×10^{9}	48.0	.147	6.71×10^{5}	.161	3.22×10^{5}	.137	.015
	1.6	724	.799	3,795	1.252	.135	2.18×10^{10}	344	.150	7.22×10^{5}	.162	3.46×10^{5}	.142	.013

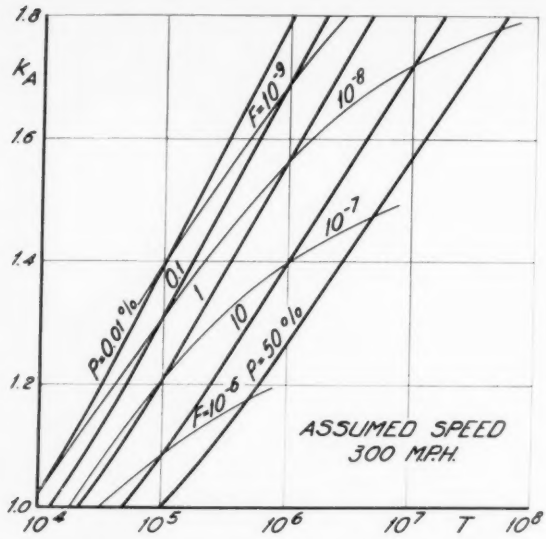


Fig. 30. Life of notched 24S-T specimens as specified under item 5 in Table I.

to be the object of special theoretical and experimental investigations. In such investigations the applicability of the assumed cumulative damage theory comes into the picture.

One important object of research in this respect is to establish the relationship between the scatter in life by life calculations based on a cumulative damage theory and *P-S-N* information on one hand and the scatter in life obtained by representative varying amplitude tests on the other.

8. Designing for Fatigue; Topic A

8.1. General

Eggwertz's formula, Eq. (45), makes it possible to present design information in forms convenient for use by the designer. In the first place there seems to be a need for design charts of two types, one more general and the other representing test results. In the general charts the various parameters of the *S-N*-function, for instance b, c and s_{aE}, are also parameters in the chart. This implies that the chart can be used for any *S-N*-function, assumed or obtained by tests and representing a mean curve or any other *P*-curve. Thus the *P*-level itself does not appear in this type of chart.

For the other type of charts the probability-level is an important parameter, whereas the *S-N* parameters do not appear. In both these two types of charts a factor characterizing the design should be used as the main variable, besides, of course, the fatigue life itself. In the charts presented in the following two sections the fictive fatigue factor, k_{Af}, (see Section 5.2) and the fatigue factor, k_A, respectively, have been chosen as design variables.

Besides these two types of charts, other generalized and more or less approximative methods can also be developed. In particular there seems to be a need for studying more in detail the influence of the relative mean load as well as of the stress concentration factor. Due to lack of test results, which are sufficiently comprehensive in a statistical sense, these two items have been studied only rather superficially in this chapter.

8.2. A General Chart for Fatigue Life

Eq. (46) of Chapter 7 contains five parameters, α, β, s_{aE} belonging to the s_a-N curve and H_0, h to the s_a-H spectrum. In order to enable the preparation of a chart for convenient evaluation of fatigue lives in miles or hours flown it is practical to base the chart on a certain "unit" value of α, which value has been chosen to $\alpha = 1{,}000$ (compare Section 6.7). Further, the basic gust curve has been used (Section 5.2) with $H_0 = H_{0b} = 3.16 \times 10^6$ and $h = h_b = 51.8\ k_{Af}$. Inserting in Eq. (46) gives the "unit life"

$$L_0 = \frac{317}{\Gamma(\beta + 1)} (51.8 k_{Af})^{\beta} e^{51.8 k_{Af} s_{aE}} \qquad (55)$$

which contains three parameters, β, s_{aE} and k_{Af}. In the general chart for this formula, Fig. 29, the upper part is applicable for $s_{aE} = 0$. As k_{Af} and L_0 are in logarithmic scales, different values of $c = 1/\beta$ give a family of straight lines. It is interesting to note that, with $s_{aE} = 0$, an increase of k_{Af} does not increase the life very rapidly if the c-value is not very low. It might, however, be observed that a change in k_{Af}, implying a change in the mean stress, usually results in a change of α, β and s_{aE}, and this normally causes a marked increase in life.

If the endurance limit is not zero, $\log L_0$ will get an increment of $51.8\ k_{Af} s_{aE} \log e$. In the lower part of Fig. 29 curves have been drawn for $s_{aE} = 0.05$, 0.10, 0.15 and 0.20 corresponding to the said increment. By measuring at a certain k_{Af}, vertically from one of these curves, or from an interpolated value between them, to a c-value in the upper part of the chart and then transferring the measured distance to the ordinate scale, the life L_0 is thus obtained with the true value of the endurance limit taken into account. It is seen that a change of k_{Af} will have a much more appreciable influence on the life when s_{aE} has a comparatively high value than when $s_{aE} = 0$.

Parallel with the L_0 scale is a scale $T_{0,300}$ which gives the life in hours if the speed is assumed to be 300 m.p.h.

The actual life of an airplane structure can finally be computed from L_0 by introducing the correct value of α as well as the H_0-value for any gust curve parallel with the basic curve, thus

$$L = 3{,}160(\alpha/H_0)L_0 \qquad (56)$$

Fig. 29 is valid primarily for $h = 51.8\ k_{Af}$. If the actual slope of the basic gust curve differs from this value, it is still possible to use the chart if k_{Af} is replaced by (see Section 5.2)

$$k_{Af}' = \frac{\log H_0}{U_1} \frac{36.1}{6.50} k_{Af} = 5.55 \frac{\log H_0}{U_1} k_{Af} \qquad (57)$$

8.3. *Fatigue Life Diagrams Based on Tests*

On the basis of some of the results presented in Table I, Figs. 30–34 have been derived, each of which shows the fatigue factor, k_A, versus the fatigue life at various probability levels, with the application of a confidence coefficient of 0.95. As it might be of some interest to see also how the failure rate is tied up with probability and fatigue life, curves have been plotted for constant failure rates. It might be emphasized, however, that it is only the critical failure rate for the whole airplane that is of fundamental significance and that it is the probability levels for the structural elements that have to be used for design purposes.

It should further be pointed out that these diagrams have been produced in the first place in order to exemplify this type of design information. Thus, Figs. 30–33 in particular should not be used without great caution for actual design, because (1) they are based on insufficiently large sample sizes, (2) the P-curves have been obtained by pooling under the non-established assumption that the P-case D were applicable, and (3) the underlying tests were run with specimens having much closer manufacturing tolerances than can be worked to in economical production. Fig. 34, for notched 75S-T, is determined on the basis of a larger sample size (387 specimens) and the results have been pooled using P-case A, which case according to Section 6.5 was fairly well established. Anyhow, also this figure is valid for the close manufacturing tolerances applied in the test series.

With these reservations, Figs. 30–32 are intended to represent elements in a wing structure subjected to gust loads. The life in hours corresponds to a speed of 300 m.p.h. The curves have been deduced by using Eggwertz's formula and applying the basic gust spectrum L in Fig. 12. It has further been assumed that $k_{Af} = k_A$—i.e., that the required ultimate static load factor is 3.75, the relative gust sensitivity factor, ν, being equal to unity and that the weight and speed have their design values. In this connection it might be emphasized that the figures cannot be used for any other value of k_{Af}, as a change in the stated factors also influences the relative mean load which necessitates a fresh calculation of the factors.

Figs. 33 and 34 are intended for pressurized cabins. It has been assumed that the load spectrum corresponds to the case where the plane climbs to an altitude at which the design pressure difference is obtained, and thereupon continues to fly at this altitude until the descent. This implies that any variations in the pressure difference that can occur en route are neglected. Additional load increments due to the effect of gusts on the cabin structure are also disregarded. Thus during each flight, it is assumed that one load cycle is applied. The failure rate curves in these two figures are valid for the assumption that the duration of flight is one hour.

In order to facilitate a picture of the overall quality of the test specimens listed in Tables I and II and to study the influence of the stress concentration factor, K_t, the life at $P = 50$ per cent and $k_A = 1.0$ has been plotted versus this factor in Fig. 35. K_t is here defined as the ratio of maximum local elastic stress to average stress in the net section of the element. Fig. 35 confirms the well-known fact that steel is a better material in fatigue than 24S-T which in turn is better than 75S-T, and the general trend of the influence of K_t is also clearly indicated by the figure.

For the evaluation of the influence of the fatigue factor, k_A and thus the relative mean load, s_m, a diagram of the type shown in Fig. 36 is useful. In this figure the ratio $L/L_{kA=1}$ is plotted versus k_A. The figure shows an interesting and rather marked trend inasmuch as it indicates that the stronger materials, as well as the specimens with lower stress concentration factors, in most cases also seem to yield a more rapid improvement by increasing k_A—i.e., by lowering the mean stress. An exception from this trend is the riveted joints in 75S-T and 24S-T. It should, however, be pointed out that these two joints were not identical, the 24S-T joint being made with loosely driven rivets for the particular purpose of finding out whether the scatter could be decreased by excluding the influence of rivets finally formed in different ways. This type of joint can also be said to represent rather bad workmanship.

It might be pointed out that it seems to be worth while, for a particular reason, to study fatigue test results in the form of diagrams of the type illustrated by Fig. 36. In order to obtain fatigue life diagrams of the type represented by Figs. 30–34, it is necessary, for attaining satisfyingly low P-levels, to apply sample sizes of more than 100 specimens not only at each of a number of different load levels but also at a number of different mean loads. In case extended research would verify consistent rules or trends as regards the course of the improvement in life ($L/L_{kA=1}$) by increase of k_A, this would mean that the testing at various mean loads could be substantially reduced, or possibly in some cases even be excluded.

8.4. *Design Procedures for Various Types of Design*

The use of design charts of the type described and exemplified in the previous section presupposes that the element probability level, P_E, is known. As indicated in Section 3.6, P_E is usually very much higher than the limit probability of failure for the whole structure, P_L, corresponding to the critical failure rate. The ratio between P_E and P_L—i.e., the probability factor c_P—varies with the type of design, in particular with the extent of redundancy. It might be possible that extended research and experience can lead to the establishment of certain approximate rules or methods, by means of which c_P can be determined or estimated for certain typical types of design. As long as this state of affairs has not been attained, design procedures of the type described in the following seem to be feasible.

In the Figs. 8 and 9 in Section 3.6, a cross section at the limit life, T_L, of the family of probability curves

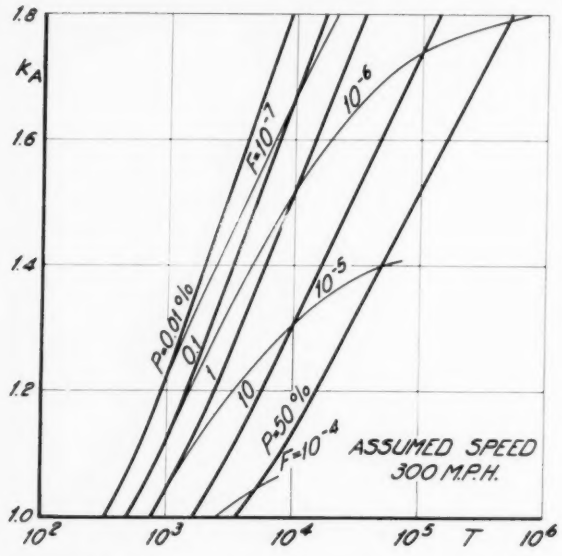


Fig. 31 Life of 75S-T riveted joints as specified under item 10 in Table I.

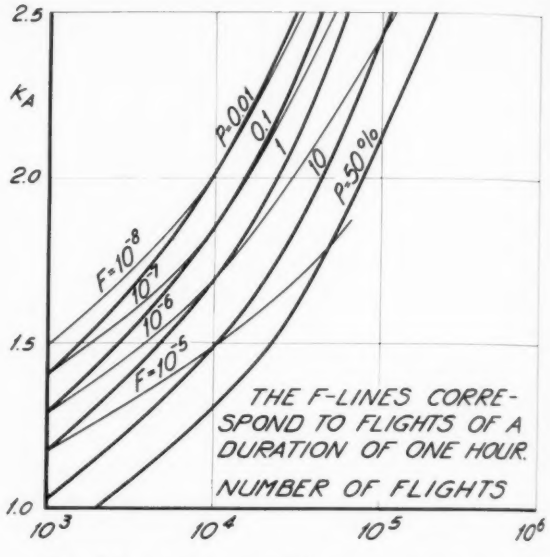


Fig. 33 Life of 75S-T riveted joints as specified under item 16 in Table I.

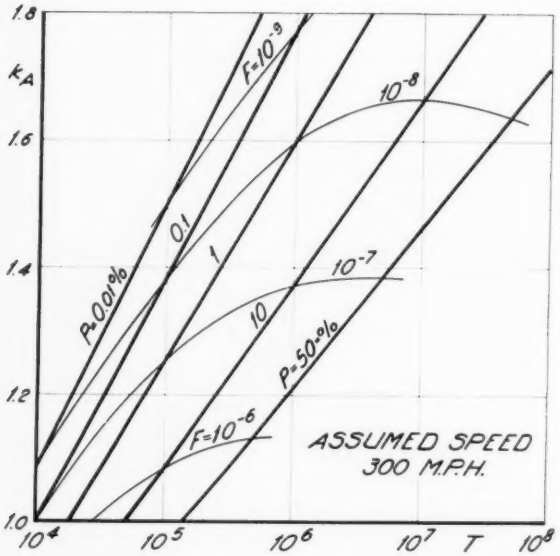


Fig. 32 Life of notched Cr-Mo steel specimens as specified under item 14 in Table I.

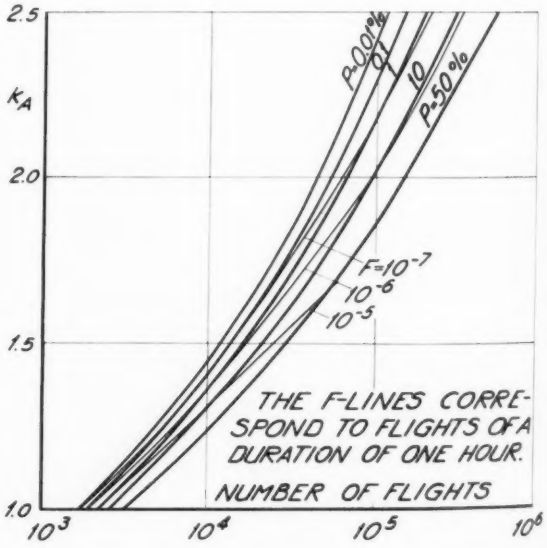


Fig. 34 Life of notched 75S-T specimens as specified under item 17 in Table I.

can be assumed to include a corresponding family of probability curves as functions of the fatigue factor, as illustrated in the schematic Fig. 37.

In Fig. 37 the "element curve" (0) corresponds to the probability of failure, P_0, for a fatigue sensitive element of the structure. For wing structures subjected to gust loads this curve can be obtained by means of the general chart given in Section 8.2 or by Eq. (46) of Chapter 7, provided that the parameters of the S-N functions have been determined for various P-levels. Should a fatigue diagram of the type shown in Section 8.3 represent, with satisfactory accuracy, the type of design intended to be used in the new airplane, then the element curve can be obtained more directly from a cross section of such a diagram at the life T_L.

If there are a number of elements, a, b . . ., which affect the fatigue life of the whole structure, it is often feasible to assume, as an initial approach, the same k_A values for all these elements. This assumption is a good first approximation, at least when all the elements are made of the same material and have about the same stress concentrations. The probabilities of failure, P_{0a}, P_{0b} . . ., may then be plotted versus k_A. If now the structure consists of m elements and it is assumed that failure in one element results in failure of the whole structure, the probability of failure for the structure can be calculated by means of simple probability laws. If it is, for example, assumed that all the m elements are identical, independent and subjected to equal loading conditions, the probability for the structure to fail is

$$P_1 = 1 - (1 - P_0)^m \tag{58}$$

corresponding to curve (1). For small values of mP_0

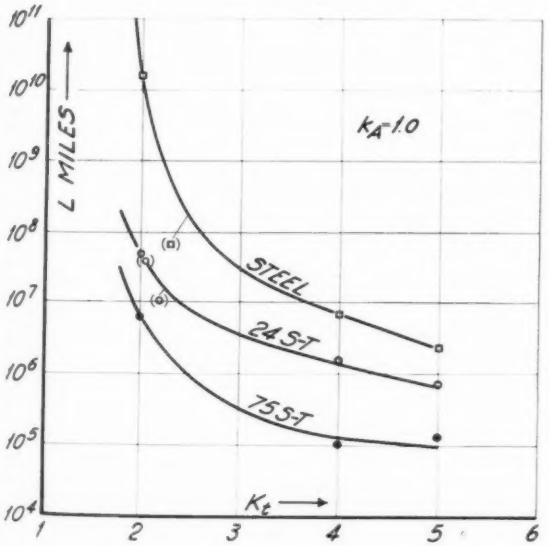


Fig. 35. Life for k_A=1.0 vs. stress concentration factor, K_t, for specimens listed in Tables I and II.

Points within brackets represent notches of a different type than the other points.

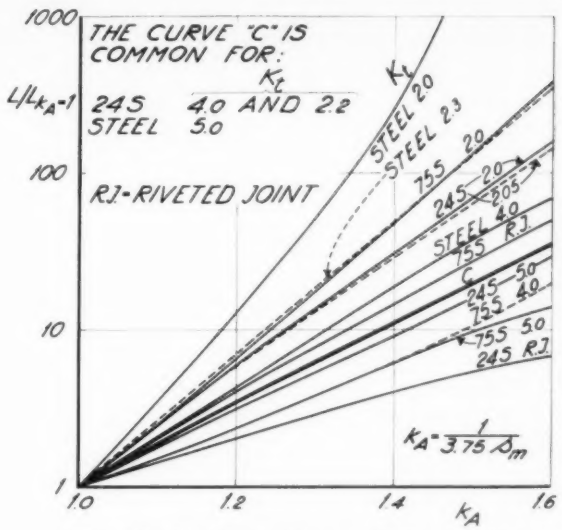


Fig. 36. Ratio between fatigue life and life at k_A=1 vs. k_A for the specimens listed in Tables I and II.

$$P_1 \approx mP_0 \tag{59}$$

It follows that this equation is usually applicable for single-spar wings, having m equally fatigue sensitive joints. In the case of a two-spar wing, failure of one spar normally implies failure of the whole wing, and this in some cases might also apply to a three-spar wing.* For these and similar cases it can thus be assumed that the structure is built up of q components

* Kennedy[53] has treated the case of a two-spar wing box under the assumption that total failure does not occur until both the spars have failed.

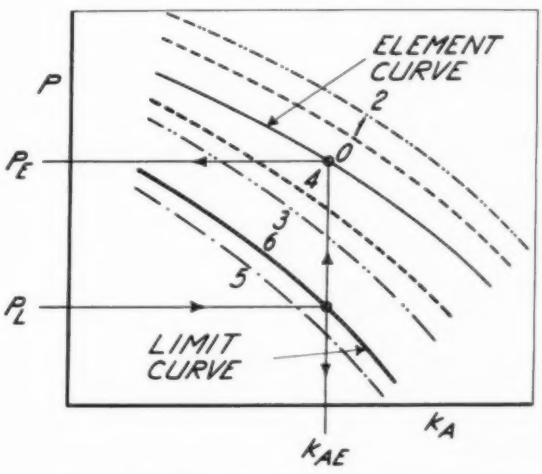


Fig. 37. Illustration of probability vs. fatigue factor at a given limit life for an element, a component and the complete structure.

(spars) and that each component consists of m elements in series. With all elements identical, the total probability of failure is

$$P_2 = 1 - (1 - P_1)^q \tag{60}$$

corresponding to curve (2), or for small values of $qP_1 = qmP_0$

$$P_2 \approx qmP_0 \tag{61}$$

In a redundant—i.e., a statically indeterminate structure—a certain number of components or elements may fail before total failure occurs. It is assumed that when at least q_c components out of q, each with m identical elements in series, have failed, the whole structure will fail. First the hypothetical case is considered when the failure of one component does not increase the loads in the other components (curve 3)

$$P_3 = \binom{q}{q_c} P_1^{q_c} (1 - P_1)^{q - q_c} \tag{62}$$

where the binomial coefficient $\binom{q}{q_c}$ is obtained from mathematical tables[11].

For small values of $(q - q_c)mP_0$

$$P_3 \approx \binom{q}{q_c} (mP_0)^{q_c} \tag{63}$$

It might be pointed out that this equation only yields the probability of failure of exactly q_c elements. As failure of more than q_c elements will also cause failure of the whole structure, the determination of the probability for the structure to fail must include the probabilities of $q_c + 1$ up to q failures. As the probability of more than q_c failures is usually small in comparison with P_3, Eq (63) may be used as an approximation.

The above assumption that failure of one component does not influence the load of the other components may in some cases be an acceptable approximation, but is usually not realistic. One way is to assume that the load that was carried before failure by the failed component is equally distributed either among all the other components or only among the adjacent ones. Normally, however, the load distribution after one failure, two failures and so on, should be determined rather accurately by experimental or theoretical methods. The failure of one component will thus yield new P-functions for all the elements of the structure, and then the P-curve (4) of Fig. 37 has to be determined. This problem is now being treated at the FFA, but it is not possible to go into details in this lecture.

Normally an airplane is regularly inspected during its life time. If it were possible to detect all cracks of a certain width at the inspections and if, further, the inspection intervals are chosen shorter than the minimum time for the development of the crack from a detectable width to complete failure of the element, the probability of fatigue failure of the entire structure would be zero. Usually it might be rather difficult to detect all cracks at an inspection. Some cracks may not be discovered until the second or third inspection after their generation. This means that the effective inspection interval is longer than the nominal one, which must be taken into account in a realistic study of the problem.

In the case of highly redundant structures, where two or more failures of components can occur before the whole structure collapses, it is obvious that even very simple inspections at which only complete failures of elements and components are discovered, will decrease the probability of failure for the structure considerably. It can be shown that the resulting probability, P_5, is approximately proportional to the factor $(t_i/T_{1,50})^{q_c - 1}$, where t_i is the inspection interval and $T_{1,50}$ is the life of a component of m elements, curve (1), corresponding to $P = 50$ per cent. As the ratio $t_i/T_{1,50}$ is usually rather small, this factor will get extremely small values when $q_c \geqq 3$. Inspection at short intervals also implies that curve (4) is no longer applicable, as the increase in load on the other components, due to the failure of one component, will only last until the failure is detected at the next inspection and the element is thereupon repaired. The reduction in probability caused by inspection should consequently be referred to a curve which is located closer to curve (3), the shorter the intervals are between effective inspections.

In Fig. 37, curve (5) represents the probability of failure for the whole structure when the inspection, for instance after each flight, is 100 per cent effective and curve (6) the same probability assuming an inspection of realistic efficiency. Curve (6) is also called the limit curve, as it gives the actual limit probability level of the structure. If now the critical failure rate, F_{cr}, is fixed, the value of P_L for the structure can be obtained from $P_L = F_{cr}T_L$. The point of intersection in Fig. 37 between the line $P = P_L$ and curve (6) gives the k_A value required, k_{AE}. Further, P_E is obtained at the intersection between $k_A = k_{AE}$ and the element curve (0).

For the design procedure thus described it was assumed that all the fatigue-sensitive elements were given the same k_A values. However, to design all the fatigue-sensitive elements with the same k_A value, k_{AE}, gives a minimum structure weight only provided that all the fatigue-sensitive elements are nominally identical and subjected to identical loading conditions. As this rather seldom is the case in actual designs, it is normally possible to reduce the weight by giving the fatigue-sensitive elements different k_A values. For such optimum analyses there are, in particular, two typical cases which have to be considered and will be briefly described here.

Should the fatigue-sensitive elements be made of the same material and have the same reduction in fatigue strength due to stress concentrations, it will be found that a weight-saving can be accomplished by applying somewhat lower k_A values for elements with a large cross-sectional area than for the smaller elements, implying a somewhat greater element probability for the heavier elements than for the lighter ones.

The other typical case to be considered is when the fatigue-sensitive elements differ considerably from each other in respect of fatigue qualities due to differences in material and notch effects. In this case it is obvious that the more fatigue-sensitive elements should be given higher k_A values than the less fatigue-sensitive ones.

Actual designs might often represent a combination of the two typical cases mentioned. Whatever type of design in this respect is used, the design procedure described above has to be somewhat modified. For the determination of the k_{AE} value of any of the fatigue-sensitive elements, it is feasible to assume either that k_A for the other elements has constant values or that their k_A values are in a certain proportion to the k_A value for the element under consideration. Obviously the procedure might have to be repeated a number of times before those k_A values for the different elements have been determined which provide the lightest structure.

In the above general description of the design procedure no regard has been given to the possibility of static failure caused by gust or manoeuvring loads of a higher severity than is normally to be expected within one or a few inspection intervals and occurring after the structure has been weakened by one or several fatigue cracks or part failures. A consideration of this matter will introduce a new curve, which, wholly or partly, might fall above the curve (6), and thus should be used as the limit curve or basis for the determination of k_{AE}.

Fig. 38 gives an example of a P-k_A diagram including the curves (0) for $T_L = 5 \times 10^4$ and 10^5 hours and the curves (1) for the same values of T_L, assuming $m = 5$ and 10. The element curves are based on test results from notched 24S-T specimens as evaluated in Fig. 30.

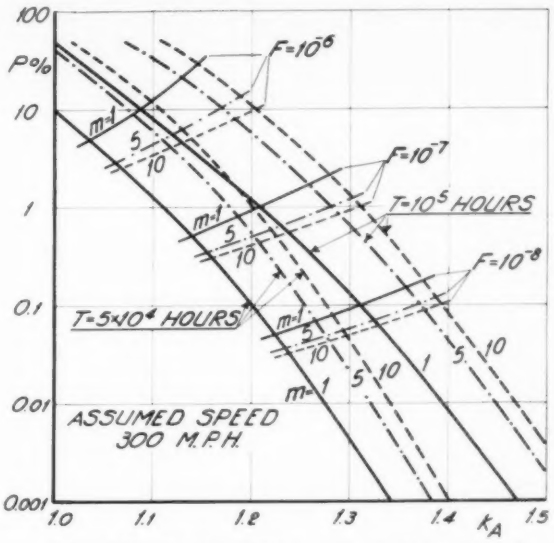


Fig. 38. Probability vs. fatigue factor for 1, 5 and 10 elements at two values of T, based on notched 24S-T specimens as specified in item 5 in Table I.

9. Checking Fatigue Properties; Topic B

9.1. *General*

The problem of checking the fatigue qualities of vital elements, such as joints, of a new type of airplane, the design of which has been completed to a stage enabling such parts to be manufactured, is of a rather different nature compared with the problem of designing for fatigue. When the design is frozen, the fatigue factor, k_A, is "built-in" into the part, which implies that this factor no longer plays a very important rôle for the designer. Thus, the purpose of checking the fatigue qualities of elements is not to establish the required value of k_A but instead to determine, as accurately as possible, the fatigue life corresponding to the element probability level, P_E. It is, of course, desirable that the check tests verify that this life of the component agrees as closely as possible with the life for which the part was designed.

The main difficulty with check testing is that for economical reasons normally only a limited number of test specimens of the vital parts in question can be tested. It is therefore of great importance that the check tests are carried out in such a way that the fatigue life corresponding to a normally rather low element probability level can be determined. This means that the limited number of test specimens should be used in such a way that not only the mean value of the fatigue life but also the standard deviation can be established with reasonable accuracy. From this point of view varying amplitude loading, corresponding to an assumed load spectrum, has the advantage, as compared with constant amplitude testing, that the whole sample of the specimens can be used for testing at one and the same loading condition. Another advantage with varying amplitude testing for the purpose of checking fatigue properties lies in subjecting elements, with the same geometry and with the same tolerances as the actual elements produced for the airplane, to loading conditions similar to those anticipated in service, which, of course, for one thing reduces the uncertainties due to the insufficiently known effects of cumulative damage.

Constant amplitude testing, however, has also certain advantages for check purposes. One is that the equipment needed is much simpler and cheaper. Furthermore it might be pointed out that there is a definite advantage in carrying out tests with the actual elements with the same type of loading, mainly of S-N type, as is used for establishing the design information on which the design of the element was based. For these two reasons it is felt that S-N testing should be used also for checking fatigue qualities, although in the future the main bulk of check testing might be made by means of varying amplitude tests. It might be added, that such constant amplitude testing with actual elements could be regarded as an important link between designing the elements, at least partly,

on *P-S-N* information and the final checking of the qualities by means of varying amplitude testing.

This chapter will only deal with check testing by means of *S-N* tests, firstly because the evaluation of results from varying amplitude tests does not offer any particular problem as such results should directly yield the determination of the fatigue qualities of the element, and secondly because published results of such tests with actual elements of airplanes, or elements representative for airplane structures, are so scarce that they are hardly possible to analyze in a more general way.

In order to profit as much as possible from the limited number of tests in case of *S-N* testing for check purposes it is obviously of great importance that the tests are carried out at such a number of load levels and at such magnitudes of these load levels that the results obtained can confirm as accurately as possible that the fatigue properties aimed at in the design stage also have been attained, or else indicate any deviations from the intended fatigue properties. The method of defining the limit life as the life to a certain low probability level, implies that the determination of at least the approximate location of the corresponding *P-S-N* curve, within a significant range of the load level, is of even greater importance than is the determination in a more exact way of the course of this *S-N* curve. Thus there is an optimum number of load levels at which a certain limited number of test specimens should be tested. This matter is worthy of special study, but as a preliminary judgment it might be suggested that when the total sample size is larger than about 20, at least three load levels should be applied. When the sample size is anywhere between 6 and 20, the test should be confined to two levels, and when the sample size is less than about 6, all the specimens should be tested at one significant load level.

9.2. *A General Chart*

From what is mentioned above it follows that in the design of general charts for the evaluation of check tests, the relative load amplitude, s_a, often is not the most suitable way of expressing the test results. Absolute values of loads or stresses can of course be used, but for general studies, applicable for various materials, "the comparative load amplitude," l_a, defined in Section 5.1, seems to have some advantages. The endurance function then takes the form

$$N = \alpha_l (l_a - l_{aE})^{-\beta} \tag{64a}$$

or

$$l_a = b_l N^{-c} + l_{aE} \tag{64b}$$

where

$$b_l = (n_D k_A/\omega_m) b \tag{65}$$

$$\alpha_l = b_l{}^{\beta} = (n_D k_A/\omega_m)^{\beta} \alpha \tag{66}$$

$$l_{aE} = (n_D k_A/\omega_m) s_{aE} \tag{67}$$

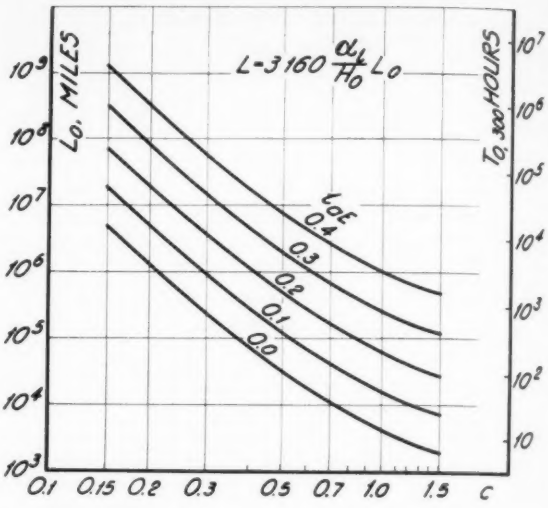


Fig. 39. General chart for checking gust fatigue life.

The value of the parameter c, and $\beta = 1/c$, is independent of in which terms the amplitude load is expressed.

Consistently the gust load curve

$$H = H_0 e^{-h_l l_a} \tag{68}$$

where

$$h_l = \frac{\omega}{0.03\ \nu v} \frac{\log H_0}{U_1 \log e} \tag{69}$$

For the basic gust spectrum

$$h_l = 13.8 \frac{\omega}{\nu v} \tag{70}$$

Eq. (46) of Chapter 7 can now be written in the form

$$L = \frac{10^6}{H_0 \Gamma(\beta + 1)} \alpha_l h_l{}^{\beta} e^{h_l l_{aE}} \tag{71}$$

If this life is related to the value $\alpha_l = 1{,}000$, and further the basic gust curve is used and $\omega/\nu v = 1$, the following unit life is obtained, which only contains the two parameters, β and l_{aE},

$$L_0 = \frac{317}{\Gamma(\beta + 1)} 13.8^{\beta} e^{13.8\, l_{aE}} \tag{72}$$

This equation has been used for the design of the general chart, Fig. 39. In comparison with the general chart in Fig. 29, Section 8.2, Fig. 39 has the advantage that the influence of the endurance limit, l_{aE}, can be directly determined.

The real life, L, in miles, is obtained by

$$L = 3{,}160 \frac{\alpha_l}{H_0} L_0 \tag{73}$$

This equation is, however, valid only for $\omega/\nu v = 1$ and for the slope of the basic gust curve. It is, of course, possible to design a more complete general

chart which includes a parameter by which the influence of variations in ω/v and the slope of the gust curve can be determined.

The parameters α_l, β and l_{aE} will normally have to be determined by check tests at three or more levels of load amplitude. They can be computed by Eqs. (38), (39), and (40), if l_a is inserted for S.

If the check tests have been limited to *two load levels*, l_{aN_1} and l_{aN_2}, and the endurance limit, l_{aE}, is assumed to be zero, the following equations are obtained from Eqs. (64b)

$$c = \frac{\log l_{aN_1} - \log l_{aN_2}}{\log N_2 - \log N_1} \tag{74}$$

$$\log b_l = \log l_{aN_2} + c \log N_2 \tag{75}$$

The life in miles is then obtained either from the curve $l_{aE} = 0$ in the general chart, Fig. 39, or from the following simplified formula

$$L = \frac{10^6}{H_0\Gamma(\beta + 1)} (b_l h_l)^\beta \tag{76}$$

9.3. *An Empirical Diagram for Two Load Levels*

With reference to Chapter 7, it seems likely that the relative load amplitude at the maximum of the damage intensity, $s_{a\,max}$, should be of particular significance for the fatigue life. In order to investigate this matter, different ways have been tried of plotting the life and other parameters for the test results condensed in Table II. It was found that if the ratio between the life and the fatigue factor, L/k_A, is plotted versus the product $k_A s_{a\,max}$, all the values fall rather consistently around a straight line as shown by Fig. 40. This fact could be utilized for the design of a diagram with a rather wide applicability, if a factor could be found which governs the spread of the points in relation to the said line. It has been found that the slope of the s_a vs. log N curve at the amplitude level $s_{a\,max}$ is significant for the spread. The said observations have led to the design of an empirical diagram at which, however, the product $k_A s_{a\,max}$ has been replaced by the product $k_A s_{a\ 2\times10^6}$. The reason for this replacement is that, although $s_{a\,max}$ only contains two S-N parameters, β and s_{aE}, it cannot be determined with good accuracy without testing at three load levels, and if three-level testing is made, the life can be directly determined by using the general chart, Fig. 39, in Section 9.2. Another reason for the replacement is that the load level corresponding to failure at 2,000,000 load cycles, has been attributed a particular significance by Walker[54] and Chilver.[28] As a measure of the slope of the s_a vs. log N curve the parameter

$$\Delta s_a = s_{a\ 5\times10^5} - s_{a\ 2\times10^6}$$

has been used. By ascribing the different values from Table II, plotted in log (L/k_A) versus $k_A\, s_{a\ 2\times10^6}$, their particular Δs_a values and by means of cross plotting, the general diagram Fig. 41 has been designed. This diagram is directly applicable for the case ν, ω and v equal to 1.0 and $n_D = 3.75$. For other values of these parameters the diagram can be used by proportioning.

In order to check the validity of this diagram all the lives listed in Table II, which are calculated exactly by Eq. (46), have been plotted in Fig. 42 versus the corresponding values obtained from Fig. 41. Fig. 42 shows that all the points are falling rather close to the 45°-line. Most of the points are within a band of ±20 per cent from the said line, which band is indicated in the figure.

It can thus be concluded that Fig. 41 might be used with sufficient accuracy for rough estimation of the life of a structural element when the two points of the S-N curve corresponding to 5×10^5 and 2×10^6 cycles to failure are known. It might be pointed out that, although Fig. 41 has been based on computations from values valid for $P = 50$ per cent, it has been found that the diagram is also applicable with good accuracy for other P-S-N curves.

Turning back to the question of the significance of a structural part being able to withstand 2×10^6 load cycles, it might only be mentioned that Fig. 41 indicates that the slope of the s_a-curve is a factor that can hardly be ignored in establishing relevant fatigue criteria. For determining this slope it is necessary to run S-N tests at least at two load levels.

Finally it might be mentioned that it is of course difficult to choose the two load levels so that failure, at a significant P-level, occurs at $N = 2 \times 10^6$ and $N = 5 \times 10^5$, respectively. If the chosen load levels give fatigue lives, which differ appreciably from these values, the diagram, Fig. 41, cannot be applied.

9.4. *An Approximate Formula and Diagram for One Load Level*

For the evaluation of test results obtained from fatigue testing at only one load level, it is feasible to give a specific value to one of the three parameters in the S-N-function and to apply a known or an assumed relationship between the two others.

When it is a question of low probability levels, notched elements and construction in light alloys, only a small error is usually introduced by putting $s_{aE} = 0$. By applying further the empirical relationship between b and c according to Eq. (41) in Section 6.7, the following equation is obtained:

$$c = \frac{-\log s_{aN_1} - \log b_0}{\log N_1 - 3.88} \tag{77}$$

where N_1 is the number of cycles corresponding to the single load level s_{aN_1}, tied up with a certain P-level.

The life in miles is found by applying Eq. (46) in Chapter 7.

$$L = \frac{7.59 \times 10^9}{H_0\Gamma(\beta + 1)} \left(\frac{h}{b_0}\right)^\beta \tag{78}$$

For "the basic gust curve" the life is

$$L_b = \frac{2{,}400}{\Gamma(\beta + 1)} \left(\frac{51.8 k_{Af}}{b_0}\right)^\beta \tag{79}$$

This equation has been used for the design of the diagram in Fig. 43, which is valid for the mean line in Fig. 25.

The life for any gust curve parallel with the basic one is obtained from

$$L = \frac{3.16 \times 10^6}{H_0} L_b \qquad (80)$$

For other slopes of the gust curve the diagram is usable by introducing k_{Af}' instead of k_{Af} according to Eq. (57) in Section 8.2.

Finally it might be mentioned that the diagram in Fig. 41 of Section 9.3 can, of course, also be used for evaluating test results from one load level, if only a certain value of Δs_a is assumed to be valid. The use of Fig. 41, however, presupposes that the chosen load level is rather close to the value giving 2×10^6 cycles at failure and at the P-level under consideration.

10. Fatigue Metering; Topic C

10.1. General

In order to study the atmospheric turbulence as well as the magnitude of loads imposed on an aircraft by gusts and manoeuvres, acceleration and strain recorders have been used since before the Second World War.[55, 56] Such instruments, until recently, were exclusively intended for *research* aims. Some types of these instruments incorporate a paper roll or a glass cylinder, on which the accelerations or the strains are recorded. By this method the course of the loads is obtained in detail but the number of man-hours required to analyse the records and transform them to gust or load histograms is considerable. For the purpose of collecting statistical data the most well-known device is the NACA V-g recorder. This apparatus was, in particular, suitable for recording the most severe loads occurring during a substantial flying time, the many lower loads not being possible to distinguish. In order to enable large scope load data, covering also the great multitude of lower load levels, to be collected in a simple way, a counting accelerometer was designed by G. Gustafsson at the FFA in 1942. The apparatus functioned fairly well, but due to our limited research capacity it was not possible to develop the instrument to such a high degree of perfection that it could be offered as a research tool for general use. A few details about this counting accelerometer have been presented in reference 26.

In Australia another counting recorder, a strain counter, was developed,[57] and this counts the number of times the strain, at the point where the device is located, exceeds a series of specific values. This strain counter is of an electronic type as it incorporates a series of switches which make contact at predetermined levels of strain, a series of relays of the thyratron type and a corresponding number of electronic counters. Another strain cycle gage with electronic counters has been developed by Jasper.[58]

In particular the continuously recording accelerometers have played an important rôle, in the first place in order to increase our knowledge of atmospheric turbulence, and this undoubtedly will remain an important research field for a considerable time to come. As it is inevitable, however, that individual airplanes operate under widely different conditions, there is also a need, and in the author's opinion a more urgent need, for *a flight instrument*, as distinct from the types of research instrument mentioned, which should be mounted on each individual airplane and record, in one way or another, the history of loads that a particular airplane has been subjected to. A "cumulative fatigue damage recorder" for this purpose was proposed by the author in the discussion following. Wills's lecture at the Second International Aeronautical Conference in May 1949 (see reference 57, pp. 397–400). This recorder was proposed to include a set or "package" of test specimens with stress concentrations representative for those of fatigue-sensitive parts of the primary structure. The intention was, further, that the specimens should be subjected in advance to various degrees of fatigue by using normal fatigue machines, so that the specimens, after having been mounted in the airplane, could be expected to fail at a series of certain percentages of the fatigue life of the airplane.

Two ways of applying the loads on such packages of specimens by using the airplane as a fatigue machine have been studied at the FFA. First the possibility was studied of bolting the package of test specimens to a member of the structure, the strains of which are representative for the loading of fatigue-sensitive parts at other places of the airplane. It was found, however, that it is extremely difficult to attach the package to the structure in such a way that the specimens really are being subjected to a load history representative for the airplane. The reason for this is that the specimens are being loaded by the elongation (or compression) of the structural member to which the recorder is attached. This implies for one thing that a rather high load in the structural member can easily cause the yield stress of the specimens to be exceeded, which would result in too low loadings of the specimens under the subsequent part of the life of the airplane. This serious drawback can, at least in principle, be overcome by a spring so arranged that it cuts off all the strains above the magnitude which would cause yielding of the specimens. The fatigue damage corresponding to such high loads would then, of course, not be felt correctly by the test specimens, but this error could possibly be either neglected or compensated by making the specimens slightly more fatigue-sensitive than critical parts of the primary structure. A more serious difficulty is the extreme care that must be taken in attaching the package of specimens to the structural member and in designing the whole recorder in such a way that all the specimens, during the whole service life of the plane, have loads applied to them of the intended levels and, furthermore, that their load levels do not change appreciably when some of the specimens have failed. All these difficulties and drawbacks with the principle of using the airplane structure as a fatigue machine for the specimens in the damage

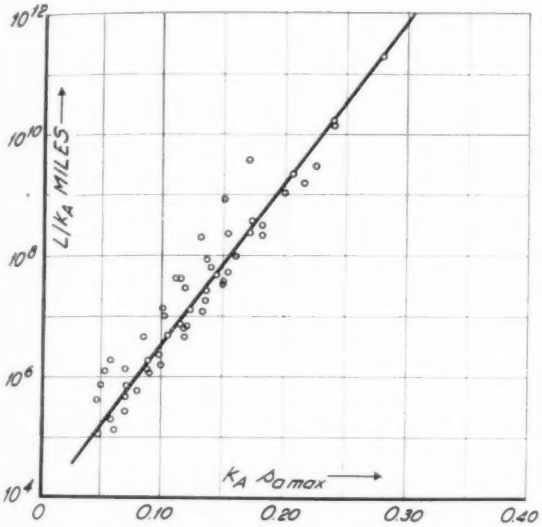


Fig. 40. L/k_A vs. $k_A\ \Delta\sigma_{max}$. Values from Table II

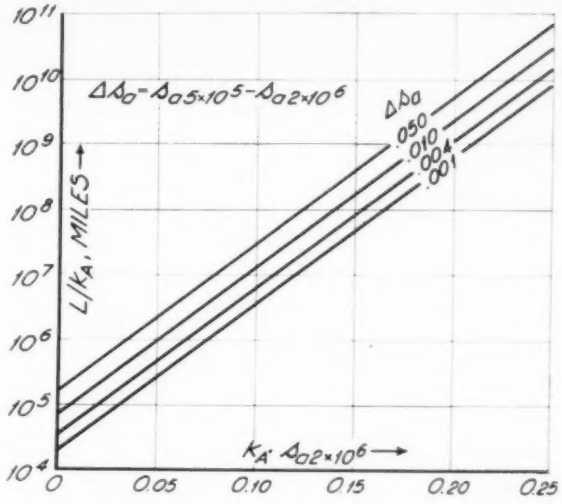


Fig. 41. Empirical diagram for checking gust fatigue life.

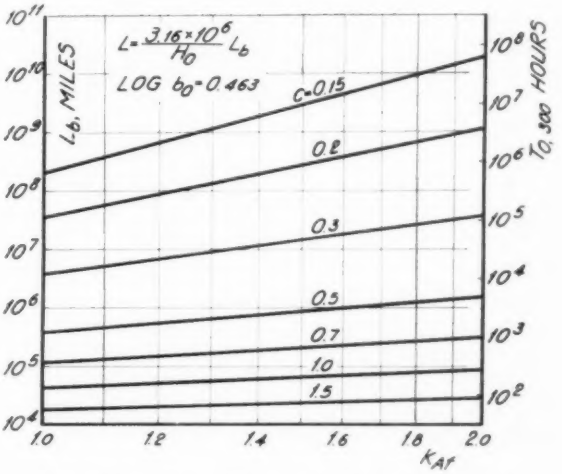


Fig. 43. Approximate diagram for checking gust fatigue life by S-N testing at one load level.

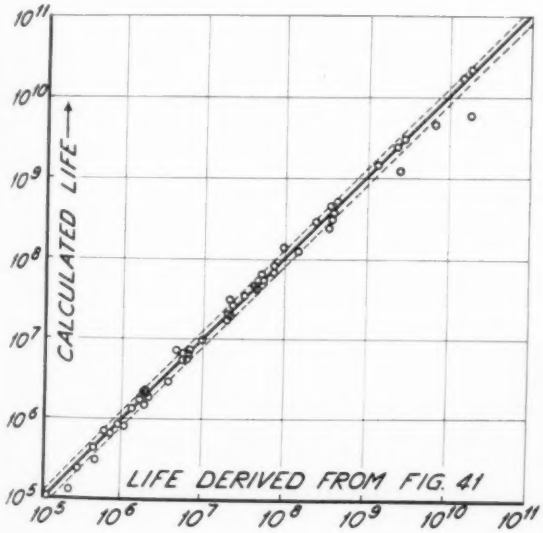


Fig. 42. Comparison between calculated fatigue life derived from Fig. 41. Dashed lines indicate deviations of 20 per cent from the mean line.

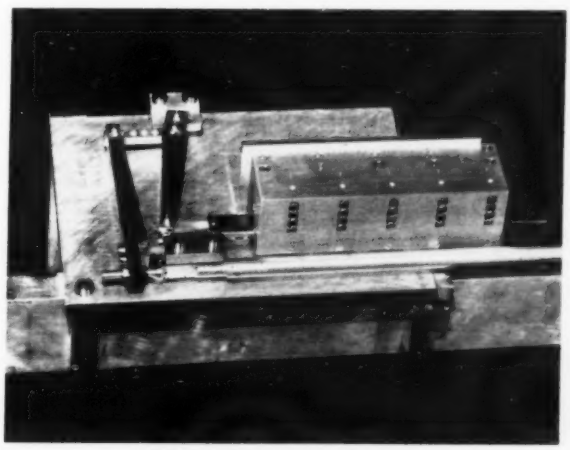


FIG. 44. The FFA strain counter.

recorder led to the idea of loading the specimens by means of the accelerations imposed on one or on a number of weights. The weights should be suspended in a balance system which magnifies the load on the weight to a level suitable for the loading of the specimens. With this method of loading, however, the apparatus becomes rather heavy, and another draw-back is that it is difficult to eliminate or compensate for the very low natural frequency of such a system. This latter difficulty can possibly be overcome by means of damping.

However, there still remains a serious difficulty with this type of fatigue damage recorder, regardless of how the specimens are loaded, and that is the great scatter in the fatigue life, that must be expected also in the specimens. This scatter can be appreciably limited by locating the stress concentrations, for instance holes, in a row along the length of the specimen, but even then the scatter will be so large that it would be necessary to have at least 5 to 10 specimens for each percentage of the limit fatigue life of the airplane. As this would imply quite a considerable total number of specimens in the recorder, this would become a rather heavy and bulky apparatus. The weight could naturally be decreased by limiting the recording to very few percentages, possibly only one, for instance at 90 per cent of the limit life. Then, however, the device would no longer be a fatigue recorder indicating the progressive damage. It would merely become a prewarning device in the more limited sense and would hardly have any obvious advantages in comparison with an efficient system of crack-detection wires.[59]

In spite of all the difficulties discussed, the author feels that continued research should be devoted to the development of fatigue damage recorders of this type, because there are a number of very marked advantages with the principle of indicating the progressive damage in a structure by subjecting to fatigue loading representative strips of the same material as used in the structure.

Regardless of the ultimate success of such research, there also seems to be good reason to apply the idea of using a simple and light counting accelerometer, not merely as a research tool but as a flight instrument, or "a fatigue meter," that in principle should be installed in every airplane. This idea was proposed by Taylor in reference 24, and a number of laboratories and instrument firms are known to have been working during the last few years on different types of counting accelerometers, some of which already are available on the market.

Regarding this type of fatigue meter, however, it might be pointed out that the fact that the airplane is not a rigid body might cause unduly large discrepancies between the "acceleration history" recorded by the accelerometer and the load history at the different fatigue-sensitive parts of the structure, for instance at various spanwise stations along the wing. This is the main reason why the author feels that recording the strains in representative parts of the structure is in principle a more relevant method of measuring fatigue damage. The above-mentioned proposals of strain recorders with electronic counters and additional electronic equipment seem, however, rather complicated. A simpler way of counting the number of strains at different levels would be to use the strain in a directly mechanical way for actuating a number of purely mechanical counters.

10.2. Mechanical, Strain Actuated Fatigue Meters

The first application at the FFA of this method was a mechanical strain counter which counts the number of strains of different levels in very much the same way as a counting accelerometer counts accelerations of different magnitudes. A model of this strain counter is shown in Fig. 44. It has been tested rather extensively in the laboratory and appears to function satisfactorily. Such an instrument could replace counting accelerometers, but as it can be mounted on almost any structural member, such as a bulkhead in a pressurized cabin or a stabilizer, a strain recorder has a wider applicability than an accelerometer. Needless to say, the instrument should preferably not be mounted on any of the fatigue-sensitive parts of the airplane, because the object of the instrument is to record the load history of the complete structural component, or a main portion thereof. The mechanical strain counter can, of course, be used both as a flight instrument and as a research tool. If it is used as a flight instrument, the strain levels to be counted could be limited to something between four and six levels. For use as a research instrument—i.e., in particular for studying the atmospheric turbulence and the dynamic response of various parts of airplanes—almost any desired number of positive and negative strain levels could be counted by incorporating a sufficient number of counters.

The possibility of another application subsequently came to the author's mind, namely the replacement of counting the strains at different levels by an integration of the corresponding fatigue damage. In such a fatigue meter, or "fatigue clock," the S-N curve for the most fatigue-sensitive part of that structural component, to which the fatigue clock is attached, is inserted or built-in into the instrument by adjusting a set of screws. The fatigue clock is not limited to a certain shape of load spectrum, as it automatically integrates the damaging effect of all the loads causing damaging strains in the structure, irrespective of the frequency distribution of the loads or strains. The sum of the fatigue damage is read off directly from one single dial. This instrument is, of course, a typical flight instrument as it continuously shows how much of the limit fatigue life is used up at any time during the operational life of the airplane.

11. Inspection and Repair; Topic D

This highly important topic will be touched upon only in two respects, which are consistent with the general scope of this paper.

The first point to be made is that inasmuch as inspection is playing such an important role in safeguarding against serious fatigue failures, the development of methods and means for efficient inspection is one of the most important research aims. In particular it seems desirable to develop automatic warning devices, such as crack-detection wires,[59,60] in order to minimize the dependence on the human factor.

In respect of repair it has been mentioned in Section 3.6 that the extent to which repairs have to be made in order to replace parts which have suffered cracks or become fractured, might impose an economical limitation on the life of the airplane. In the first place this possibility has to be considered for highly redundant structures which are designed with a high element probability, P_E.

In order to enable a computation of the repair rate, defined as the number of replacements during a certain time interval, it is feasible to study the average number of failures of structural elements in relation to time. When the element curve (0) (see Sections 3.6 and 8.4) is known for all fatigue-sensitive elements, a, b, c, . . . in the structure, the average number of element failures is obtained as

$$M = P_{0a} + P_{0b} + P_{0c} + \ldots \qquad (81)$$

In the case where all the elements are identical and independent, this equation takes the form

$$M = mqP_0 \qquad (82)$$

where

mq = the total number of identical elements
P_0 = the probability of failure during the time T for one element

Fig. 45 shows the average number of element failures versus life in hours for a structure consisting of various numbers of identical elements. The diagram is based on notched 24S-T specimens as specified in item 5, Table I, and is valid for an assumed speed of 300 m.p.h.

From diagrams of this type the repair rate, expressed in for instance number of repairs per 100 hours of flight, can be evaluated. It is, however, not the number of repairs that is the governing factor but instead the cost of the repairs during a certain time interval. This only implies a broader definition of the concept of repair rate. The judgment of what could be considered a maximum tolerable repair rate in this wider meaning lies, of course, primarily with the operator, although it naturally is a question that should be taken into account when designing the airplane.

12. Fatigue Requirements

Both the ICAO Standards and the civil air regulations in some countries have contained requirements or recommendations regarding fatigue for about nine years. During this time they have undergone slight modifications but basically they have had essentially

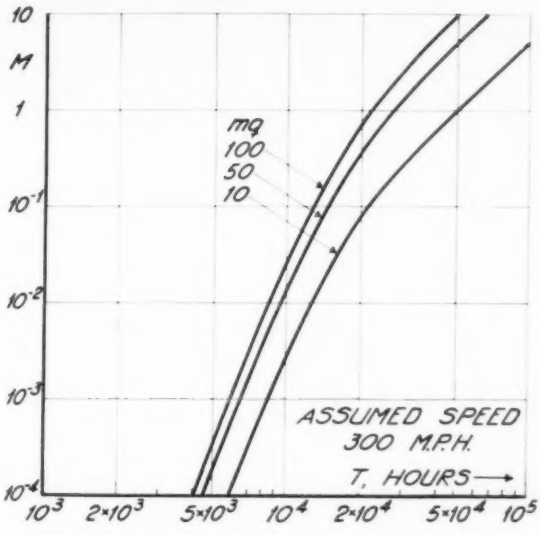


Fig. 45. Average number of element failures vs. life in hours for a structure consisting of 10, 50 and 100 identical elements. Diagram based on notched 24S-T specimens as specified in item 5 in Table I.

the same scope and implication as that of the requirements valid to-day.

The current ICAO Standards[14] are formulated as follows:

"3.11.—Fatigue Strength

The strength and fabrication of the aeroplane shall be such as to ensure that the probability of disastrous fatigue failure of the aeroplane structure under repeated loads anticipated in operation is extremely remote. Where a type of construction is used for which experience is not available to show that compliance with the other Standards of Chapter 3 will ensure the strength of the structure under repeated loads, its strength under such loads shall be substantiated so far as practicable by suitable investigations.

Note 1. With certain types of structure, incipient or progressive fatigue failures can be readily detected by inspection in time to avoid disastrous fatigue failure or serious reduction in strength.

Note 2. Fatigue tests on parts of the structure where high stress concentration is suspected will assist in showing satisfactory fatigue strength. The magnitude, and the number of repetitions of the loads resulting from gusts, manoeuvres, ground loads and vibration, will need to be assumed."

Obviously these Standards can hardly be considered to be of assistance to the designer as they are only given in qualitative terms. This is, however, in a way understandable, as our knowledge about all the complex aspects of aeronautical fatigue is still so inadequate. Although this was even more the case some eight years ago, the author, at the Second Session of the PICAO Air-Worthiness Division, 1947, made a rather specific proposal of formulating the fatigue clause in the Standards in an explicit and quantitatively determined way.[3a] The implication of this proposal was that a "limit life" should be established for critical components of the primary structure and that fatigue factors of safety

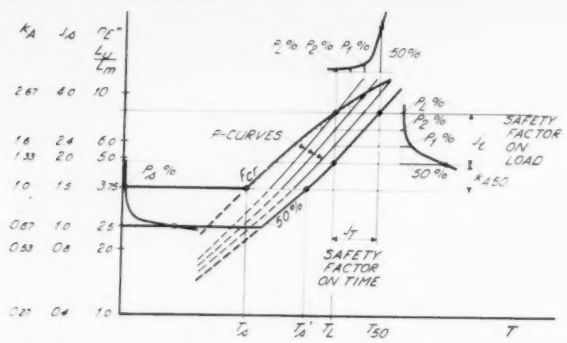


Fig. 46 Schematic illustration of static strength and factors of safety for fatigue, for a statically determinate structure with one fatigue-sensitive element

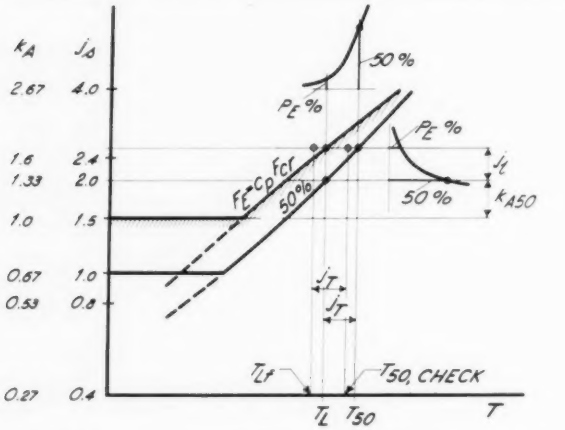


Fig. 47 Schematic illustration of static strength and factors of safety for fatigue, for an element in a redundant structure.

should be applied to ensure that fatigue failure of such a component were prevented during this limit life. Two factors of safety were proposed, one on the magnitude of the applied loads and the other in the number of load cycles—i.e., on lifetime. It was suggested that the order of magnitude of the two fatigue factors should be 1.2 for the load-based factor and 2 to 3 for the time-based factor and that the exact magnitude of the two factors might depend upon the reliability of available information as regards load spectrum and variation in the fatigue qualities of the structural parts as well as the extent to which fatigue tests are made. The proposal was not agreed upon, which, however, was not a big surprise in view of the lack of knowledge about the fatigue problem and considering that the interest in aeronautical fatigue was rather moderate at that time.

During the eight years which have elapsed since the said proposal was made, an enormous amount of research has been conducted on all the main aspects of fatigue in aeronautics. In particular the statistical nature of both the applied loads and the fatigue properties of structures has been explored and better understood. We should therefore be in a much better position to-day to make a sound judgment about the questions (1) whether or not official specific fatigue requirements or recommendations ought to be made, and (2) the terms in which such requirements should be expressed. The last item will be discussed first because it is only if appropriate terms for expressing the requirements can be found that there are any prospects of getting a more general approval of proposals for official requirements, be they of a national or an international scope.

To begin with a few words might be said about the concept of factors of safety for fatigue. Besides the proposal reviewed above, a number of similar suggestions have been made lately regarding such factors of safety. Walker[61] has proposed a factor of 1.25 on the alternating load—i.e., on the gust spectrum—and a factor of 1.5 on life, the latter factor being defined by stating that the "safe minimum" of the cycles should be calculated as $^2/_3$ of the logarithmic mean value of the endurance for a number of tested components. Shanley[8] has advocated the use of a load-based factor of safety for fatigue of 1.25.

Superficially seen there appear to be some good reasons why the safety against fatigue failures should be provided by a load-based factor of safety. Firstly, such a factor would seemingly be more consistent with the load-based factor of safety, 1.5, on the static limit load, and its numerical value would be of the same order of magnitude. Secondly, it has been observed by many scientists that the scatter in stress at S-N testing is very much smaller than the scatter in number of load cycles, which implies that the scatter would seemingly be easier to cover by a load-based factor than by a time-based one. On the other hand there are some definite advantages in applying a fatigue factor on time due to the fact that life is the significant concept for fatigue as was emphasized in Chapter 3.

In order to clarify the question of whether the safeguard against fatigue failure should be established by a factor of safety on load or on life (or else be provided in any other way), it is necessary to decide on a criterion for a correct or at least appropriate definition of such factors. If the concept of the critical failure rate is agreed upon as the basis for safety considerations in respect of fatigue, it seems logical to use the failure rate also as the criterion for judging the feasibility of fatigue factors of safety, defined in different ways. This means that the definition of a factor is in principle more appropriate the closer a certain numerical value of the factor can yield the same maximum failure rate for different types of design and different limit lives.

The relationship between the critical failure rate, various levels of probability of failure and some factors of safety is illustrated by the schematic diagram, Fig. 46, which corresponds to the type of chart exemplified in Figs. 30–32. Fig. 46 thus shows the static strength as a function of lifetime for the simple case of a statically determinate wing structure having only one fatigue-sensitive element, if it is furthermore assumed that no inspection is made. The static strength is expressed simultaneously in terms of the fatigue factor, k_A, the

static factor of safety, j_s, and the element load factor, n_E. The shaded heavy curve indicates the static strength, or structural area, required in order to comply with both the static load requirements and fatigue life demands, the latter being expressed in terms of the maximum permitted or critical failure rate, F_{cr}. Up to the ("static") life T_s the static requirements determine the cross-sectional structural area necessary. For longer lifetime fatigue considerations necessitate an increase of the area. This increase can be divided into two parts, k_{A50} and j_l, as marked at the vertical intersection of the curves at an assumed limit life, T_L. The part defined by the factor k_{A50} is a measure of the extent by which the area has to be increased if there were no scatter, in which case the fatigue qualities could be represented by the curve for 50 per cent probability of failure. The factor k_{A50} is thus not a factor of safety (compare Section 3.2); it should rather be considered as representing a "limit" condition, which in a way corresponds to the limit static load.

The other part, j_l, of the increase in area necessary to provide a certain limit fatigue life with a desired certainty has the nature of a factor of safety corresponding to the factor of safety on top of the static limit load. This fatigue factor of safety is thus the ratio

$$j_l = k_{AFcr} / k_{A50} \tag{83}$$

It should be observed that for a life shorter than T_s' (Fig. 46)—i.e., the life corresponding to $k_{A50} = 1$—this factor of safety is greater than the increase in area (on top of that determined by static requirements) which is necessary for ascertaining a safe fatigue life with a maximum critical failure rate. Between the lives T_s and T_s' this increase in area is determined directly as k_{AFcr}.

The corresponding factor of safety on time is accordingly defined as the ratio

$$j_T = T_{50} / T_L \tag{84}$$

where T_{50} is equal to the "design" fatigue life, T_D, when the total scatter is intended to be covered by j_T (compare Section 3.5).

Besides the 50 per cent curve a few more P-curves have been drawn in Fig. 46 in order to clarify the relationships between probability of failure, critical failure rate and the two alternative factors of safety, j_l and j_T. Schematic distribution curves are further presented, corresponding to the scatter bands in load and time. For comparison a distribution curve is also shown for the static strength to the left of the figure. This distribution curve, however, must be regarded as rather hypothetical, because the static factor of safety of 1.5 has never been clearly defined as a pure "scatter factor" covering the scatter and other uncertainties between limit and ultimate load down to a certain small probability of static failure (P_s).

Regarding the load-based factor of safety, it might be pointed out that such a factor can be defined in at least the three following ways:

(a) the factor, j_{la}, is applied on both the amplitude load and on the mean load of the load spectrum.

(b) the factor, j_{lb}, is applied merely on the amplitude load of the load spectrum.

(c) the factor, j_{lc}, is applied on the amplitude load of the S-N curve by dividing the ordinates by the factor.

A closer study of these definitions reveals that one and the same numerical value of these factors yields different magnitudes of k_A—i.e., different structural areas—and thus also different lifetimes. Only the factor j_{la} is directly comparable to the fatigue factor k_A and is, in fact, identical to j_l in Fig. 46. The difference in results when using j_{lb} as compared with those obtained by j_{la}, depends on the extent to which the s_a-N curve is altered by a change of the mean load. The difference could, of course, be compensated by giving the factor j_{lb} a greater numerical value, but in any case it seems preferable to define the factor as j_{la}, because there seems to be little reason to assume that the relative mean load would be subjected to appreciably less uncertainty than the relative amplitude loads, at least for the case where the spectrum of applied loads is mainly assumed to be a given quantity as advocated in Section 3.6.

As regards applying a factor, j_{lc}, on the mean S-N curve, this might seem advantageous as it would directly cover a certain desired part of the comparatively limited scatter in S with S-N results. However, this method of applying a factor of safety does not seem appropriate, firstly because it would more easily lead to the neglect of the other uncertainties besides the scatter in the S-N qualities, secondly because such a factor would cover a certain P-range only for the case of the P-case D and it would therefore be rather difficult to ascribe this factor an appropriate value for other P-cases, and thirdly because a factor j_{lc} would be difficult to apply in varying amplitude testing. It thus seems that a factory of safety on load should be defined as j_{la}—i.e., j_l in Fig. 46 and Eq. (83).

Returning now to the question of whether the fatigue factor should be applied on load or on time, Fig. 46, as well as Figs. 30–32, immediately indicate that neither of the two factors, if they have given constant values, can yield a certain failure rate, as the curves for F_{cr} and $P = 50$ per cent are not parallel in a logarithmic plot. This implies that for the simple case of a statically determinate structure, embodying one fatigue-sensitive joint of a certain type and of a certain material, a numerically prescribed factor of safety can provide a certain required level of safety at the most at one single value of the limit life. For all other lifetimes the level of safety would be either inadequate or unnecessarily high, implying a too heavy structure weight.

The variation in failure rate due to a constant value of either a load-based or a time-based factor of safety would be of moderate magnitude, and could, thus, be tolerable, if all transport airplanes were built of the same material, in a statically determinate way and with about the same stress concentrations in the fatigue-

sensitive parts. This is, however, not the case and it is easily found that different types of materials, different structural detail designs and different degrees of redundancy affect the failure rate to an extremely large extent, if only one value for the factor of safety were to be applied. For instance, a comparative study of Figs. 30, 31 and 32 reveals that merely a change of material together with moderate changes in stress concentrations necessitate quite substantial changes of the value of the factor of safety in order to provide a certain failure rate, for instance 10^{-9}. However, a still more pronounced effect on the magnitude of the factor of safety is caused by adopting a redundant type of construction and by inspection, and the necessary magnitude of the factor is, of course, also dependent on the degree of redundancy and the effectiveness of the inspections.

In general it can be stated that the factor of safety could be ascribed a smaller value the more pronounced the redundancy is and the more effective the inspection is. This is illustrated by comparing Fig. 46 with Fig. 47, which shows that the failure rate for an element of a redundant structure, $F_E = c_P F_{cr}$, determines the structural area required. It should be observed that F_E varies with time.

From the above discussion it should be obvious that no single value of a factor of safety, be it load-based or time-based, can yield a failure rate or level of safety that is even of the same order of magnitude for different limit lifetimes and different types of material, structural designs and inspection procedures. It therefore does not seem rational to prescribe a certain numerical factor of safety in airworthiness requirements. As, furthermore a multitude of factors of safety, for different materials and designs, etc., would be highly unpractical, the "factor-method" does not seem to be the correct solution for use in official requirements. Instead it is proposed that a specified value of the critical failure rate should be entered into such requirements. In principle the only factual change that would then have to be made in the present ICAO Standards would be to replace the words "extremely remote" with a specified figure. The determination of this figure is a matter that should appropriately be handled by relevant technical committees in the various countries and in ICAO, but it is believed that the figure suggested above of a maximum critical failure rate of 1 in 10^9 hours of flight is of the right order of magnitude.

Concerning the question of whether any numerically specified rule should at all be formulated in airworthiness regulations, it might be stated, in the first place, that the requirement proposed above would have the advantage (also compared with most requirements regarding other questions of safety) that it would ascertain a uniform, clearly defined and sufficiently high level of safety without prescribing in detail how this should be attained. Secondly it might be emphasized that it does not seem advisable, in view of all the serious fatigue failures in recent years, to have no specific rules for this particular subject of safety.

Having arrived at the conclusion that prescribing a certain maximum failure rate is the most practical way of expressing fatigue requirements, a further discussion of whether, using the factor-method, the factor of safety should be applied on load or on time seems to be of little consequence. It must be emphasized, however, that the proposal to use the critical failure rate as the only design criterion presupposes that all uncertainties in all the aspects of aeronautical fatigue are treated statistically. If any uncertainties are omitted from the statistical analysis, they should either be covered by conservative assumptions about each particular item (compare Section 3.6) or be covered by appropriate factors when the analysis has been completed. The first alternative is in most cases preferable, but if the factor-method is used, applying the factor on time, instead of on load, seems to be more appropriate, firstly because fatigue is a question of lifetime and secondly because the scatter, in particular in the case of varying amplitude tests, is recorded in number of cycles, and can thus be covered directly by a time-based factor. Raising the load spectrum by a factor on load—i.e., changing the slope of the load spectrum curve (see Fig. 12)—appears to be less realistic.

In this connection an interesting observation might finally be made. As stated, the main difficulty with the important topic of check testing is the small sample size that can normally be applied. Although the mean values of such tests, be they of the constant or the varying amplitude type, can be determined with a fair degree of accuracy, the scatter band [defined as the product $k_p s_l$, Eq. (32)] will be very large, if the coefficient of confidence has to be reasonably high, due to the limited sample size, as is illustrated by Figs. 13 and 14 in Section 6.2. A thorough statistical fatigue analysis of the structure during the design stage seems, however, to enable a full utilization of such an analysis in order to assess the finally permitted limit life from check tests on the vital fatigue-sensitive elements. With reference to Fig. 47, this method would imply that, if the mean life, denoted $T_{50,\,\text{check}}$, were obtained by check tests, the finally permitted limit life, $T_{L,f}$, would be obtained by dividing $T_{50,\,\text{check}}$ with the factor $j_T = T_{50}/T_L$ as obtained by the said statistical calculations at the design stage. This procedure obviously implies a bridge between designing for fatigue (Topic A) and check testing (Topic B). It should, of course, be observed that applying the said factor j_T does not mean using the factor-method, the procedure is merely a method for compliance with the failure rate criterion.

A corresponding procedure can evidently be used by applying, in principle, the corresponding factor on load, j_l, (see Fig. 47) instead of j_T. This would have the advantage, in particular for varying amplitude testing, that the load level would be raised resulting in a shorter testing time but, on the other hand, applying the factor on load might introduce some errors or uncertainties as has been explained above.

Regardless of how such a factor is applied, this method of taking full advantage of all the fatigue calcu-

lations carried out at the design stage, which calculations should be based on rather reliable, general design information when such becomes available, should possibly be referred to in the fatigue clause of airworthiness requirements. A recommendation could thus be inserted implying that, in showing compliance with the failure rate criterion by means of check testing, the designer might support the calculation of the final limit life by the statistical fatigue analysis performed in designing the airplane. This obviously means that a thorough analysis at the design stage, based on relevant design information, would be directly beneficial for the designer in terms of permitted limit life of his product.

13. Concluding Remarks

In the author's opinion the fatigue problem differs from most other problems in aeronautics because of the great scatter, which is to a large extent of an inherent nature and is therefore inevitable. It thus seems natural to use the tool we have for dealing with unknown factors in a rather accurate way, and that is by statistical mathematics. This has, of course, been done by many scientists, not in the least in the U.S.A., but has mainly been confined to a statistical treatment of the scatter in the endurance properties of specimens tested in laboratories.

In this paper the outlines of a general policy have been drawn up, by which the whole problem of fatigue in fixed wing aircraft can be grasped by means of statistics. Furthermore, some methods have been presented enabling a treatment of most of the detailed problems in an analytical way. It should be emphasized, however, that within the frame of the picture of aeronautical fatigue thus outlined, there are quite a few obscure or missing bits of the puzzle that have to be clarified or put in place before the picture itself stands out clearly. To disperse the obscurity in a reasonable time, research must be conducted to such a large extent and at such a rate that practically only the big countries, in the first place the United States and Great Britain with their well-equipped laboratories, are able to make the necessary progress. I feel very strongly that the contribution that can be made by the smaller laboratories, such as the FFA, is very limited. However, as the field of fatigue lends itself so well to joint effort, we at the FFA will be happy to co-operate in any way that may be suggested.

14. Acknowledgments

The honoring invitation, in April, 1954, to present this lecture gave me reason to carry through, in a limited space of time, theoretical investigations along certain lines of thought that have been in my mind for some years. To accomplish this, a team was formed, consisting of Professor Emeritus W. Weibull, consultant to the FFA, Mr. G. Wållgren, Head of our Structures Department, Mr. S. Eggwertz and Mr. S. Alfredson of the same Department, and myself. The skill, enthusiasm and energy with which my colleagues have assisted me in the work has been outstanding. For this I wish to express my deep appreciation.

I also wish to express my thanks to Mr. B. Noton for his help on the linguistic side, to Mrs. B. Hoffström for the preparation of the ink figures and to my secretary, Mrs. M. Lagergren, for carefully typing the manuscript.

Upon my arrival in the U.S.A. towards the end of November, it remained for me to prepare a condensed version of the lecture suitable for the verbal presentation. This was made possible thanks to the generous secretarial assistance afforded to me by the Institute of the Aeronautical Sciences.

References

[1] Kaul, H. W., *Statische Erhebungen über Betriebsbeanspruchungen von Flugzeugflugeln*, pp. 307–313, 1938 Jahrbuch der Deutschen Luftfahrtforschung, Lilienthal-Gesellschaft für Luftfahrtforschung, Erganzungsband.

[2] Bland, R. B., and Sandorff, P. E., *The Control of Life Expectancy in Airplane Structures*, Aeronautical Engineering Review, Vol. 2, No. 8, pp. 7–21, August, 1943.

[3a] Lundberg, B., *Note on Fatigue Strength Requirements for Aircraft*, Provisional International Civil Aviation Organization Airworthiness Division, Second Session, Doc. 2936, AIR/147, March 13, 1947. Also published as Technical Note No. HE-206 of the Aeronautical Research Institute of Sweden (FFA), Stockholm, 1947.

[3b] Lundberg, B., *"Bear-Up" Requirements for Aircraft*, Aero Digest, Vol. 55, No. 6, pp. 56–58 and 120–122, December, 1947. (A Condensed Version of Reference 3a.)

[4] Rhode, R. V., *Some Observations on the Problem of Fatigue of Aeroplane Structures*, Fourth Anglo-American Aeronautical Conference, 1953.

[5] Walker, P. B., *The Fatigue Situation for Civil Aircraft*, Royal Aircraft Establishment, Report Structures 129, May, 1952.

[6] Gumbel, E. J., and Carlson, P. G., *Extreme Values in Aeronautics*, Journal of the Aeronautical Sciences, Vol. 21, No. 6, pp. 389–398, June, 1954.

[7] Accident Investigation Report. Civil Aeronautics Board, SA-271, Released: May 27, 1954.

[8] Shanley, F. R., *Fatigue Analysis of Aircraft Structures*, The RAND Corporation, U. S. Air Force Project RAND, RM-1127, July, 1953.

[9] Mangurian, G. N., *Is the Present Aircraft Structural Factor of Safety Realistic?*, Aeronautical Engineering Review, Vol. 13, No. 9, pp. 63–75, September, 1954.

[10] Cramér, H., *Mathematical Methods of Statistics*, Uppsala, 1945.

[11] Hald, A., *Statistical Tables and Formulas*, New York and London, 1952.

[12] Tye, W., *Modern Trends in Civil Airworthiness Requirements*, The Journal of the Royal Aeronautical Society, Vol. 56, No. 494, pp. 73–108, February, 1952.

[13] Report of the Council to the Assembly on the Activities of the Organization in 1953, International Civil Aviation Organization, Doc. 7456, A8-P/2, p. 2, June, 1954.

[14] Airworthiness of Aircraft, Annex 8 to the Convention on International Civil Aviation. International Civil Aviation Organization, April, 1952.

[15] Airplane Airworthiness—Transport Categories, Pt. 4b of Civil Air Regulations, Civil Aeronautics Board, U.S. Department of Commerce, December 31, 1953.

[16] Pratt, K. G., *A Revised Formula for the Calculation of Gust Loads*, NACA TN 2964, June, 1953.

[17] Lundberg, B., and Wållgren, G. G. E., *A Study of Some Factors Affecting the Fatigue Life of Aircraft Parts with Application to Structural Elements of 24S-T and 75S-T Aluminum Alloys*

The Aeronautical Research Institute of Sweden (FFA), Report No. 30, October, 1949.

[18] Dryden, H. L., Rhode, R. V., and Kuhn, P., *The Fatigue Problem in Airplane Structures.* Fatigue and Fracture of Metals. A Symposium held at The Massachusetts Institute of Technology, pp. 18–51, June 19–22, 1950.

[19] Fung, Y. C., *Statistical Aspects of Dynamic Loads*, Institute of the Aeronautical Sciences, Preprint No. 376, July, 1952.

[20] Donely, P., *Summary of Information Relating to Gust Loads on Airplanes*, NACA TR 997, 1950.

[21] Walker, W. G., and Steiner, R., *Summary of Acceleration and Air-Speed Data from Commercial Transport Airplanes during the Period from 1933 to 1945*, NACA TN 2625, February, 1952.

[22] Walker, W. G., *Summary of Revised Gust-Velocity Data Obtained from V-G Records Taken on Civil Transport Airplanes from 1933 to 1950*, NACA TN 3041, November, 1953.

[23] Rhode, R. V., and Donely, P., *Frequency of Occurrence of Atmospheric Gusts and of Related Loads on Airplane Structures*, NACA Wartime Report, Advance Restricted Report L4I21, November, 1944.

[24] Taylor, J., *Measurement of Gust Loads in Aircraft*, Journal of the Royal Aeronautical Society, Vol. 57, No. 506, pp. 78–88, February 1953.

[25] Press, H., and McDougal, R. L., *The Gust and Gust-Load Experience of a Twin-Engine Low-Altitude Transport Airplane in Operation on a Northern Transcontinental Route*, NACA TN 2663, April, 1952.

[26] Hislop, G. S., *Clear Air Turbulence over Europe*, The Journal of the Royal Aeronautical Society, Vol. 55, No. 484, pp. 185–225, April, 1951.

[27] Williams, K., *Fatigue Life of Wing Components for Civil Aircraft*, The Journal of the Royal Aeronautical Society, Vol. 56, No. 503, pp. 842–848, November, 1952.

[28] Chilver, A. H., *The Estimation of Fatigue Damage in Aircraft Wing Structures*, Journal of the Royal Aeronautical Society, Vol. 58, No. 522, pp. 396–402, June, 1954.

[29] Weibull, W., *A Statistical Representation of Fatigue Failures in Solids*, Kungl. Tekniska Högskolans Handlingar (Transactions of the Royal Institute of Technology, Sweden), No. 27, 1949.

[30] Weibull, W., *A Statistical Distribution Function of Wide Applicability*, Journal of Applied Mechanics, Vol. 18, No. 3, pp. 293–297, September, 1951.

[31] Hald, A., *Statistical Theory with Engineering Applications*, New York and London, 1952.

[32] Wallis, W. A., *Use of Variables in Acceptance Inspection for Percent Defective*, Techniques of Statistical Analysis, Statistical Research Group, Chapter 1, Columbia University, New York and London, 1947.

[33] Schott, G. J., *The Statistical Significance of a Few Fatigue Results*, National Aeronautical Establishment, Laboratory Report LR-58, May, 1953.

[34] Palmgren, A., *Die Lebensdauer von Kugellagern*, Zeitschrift des Vereins deutscher Ingenieure, Vol. 68, No. 14, pp. 339–341, April, 1924.

[35] Weibull, W., *A New Method for the Statistical Treatment of Fatigue Data*, Saab Aircraft Company, Technical Note 30.

[36] Head, A. K., *Statistical Properties of Fatigue Data on 24S-T Aluminium Alloys*, Aeronautical Research Laboratories, Structures and Materials Note 180, February, 1950.

[37] Weibull, W., *Static and Fatigue Properties of Threaded Bolts*, The Aeronautical Research Institute of Sweden (FFA). To be published in the FFA Report Series.

[38] Ravilly, E., *Contribution à l'Etude de la Rupture des Fils Métalliques Soumis à des Torsions Alternées*, Publications Scientifiques et Techniques du Ministère de l'Air, No. 120, 1938.

[39] Grover, H. J., Bishop, S. M., and Jackson, L. R., *Fatigue Strengths of Aircraft Materials Axial-Load Fatigue Tests on Notched Sheet Specimens of 24S-T3 and 75S-T6 Aluminum Alloys and of SAE 4130 Steel with Stress-Concentration Factors of 2.0 and 4.0*, NACA TN 2389, June, 1951.

[40] Grover, H. J., Bishop, S. M., and Jackson, L. R., *Fatigue Strengths of Aircraft Materials Axial-Load Fatigue Tests on Notched Sheet Specimens of 24S-T3 and 75S-T6 Aluminum Alloys and of SAE 4130 Steel with Stress-Concentration Factor of 5.0*, NACA TN 2390, June, 1951.

[41] Luthander, S., and Wållgren, G., *Bestämning av livslängden vid utmattningsprov med variabla lastgränser*, The Aeronautical Research Institute of Sweden (FFA), Report No. 18, 1946.

[42] Wållgren, G., *Fatigue Tests with Stress Cycles of Varying Amplitude*, The Aeronautical Research Institute of Sweden (FFA), Report No. 28, 1949.

[43] Wållgren, G., *Direct Fatigue Tests with Tensile and Compressive Mean Stresses on 24S-T Aluminium Plain Specimens and Specimens Notched by a Drilled Hole*, The Aeronautical Research Institute of Sweden (FFA), Report No. 48, 1953.

[44] Dolan, T. J., Richart, F. E., and Work, C. E., *The Influence of Fluctuations in Stress Amplitude on the Fatigue of Metals*, ASTM Bulletin, Vol. 49, pp. 646–682, 1949.

[45] Newmark, N. M., *A Review of Cumulative Damage in Fatigue. Fatigue and Fracture of Metals.* A Symposium held at the Massachusetts Institute of Technology, pp. 197–228, June 19–22, 1950.

[46] Hardrath, H. F., and Utley, E. C., Jr., *An Experimental Investigation of the Behavior of 24S-T4 Aluminum Alloy Subjected to Repeated Stresses of Constant and Varying Amplitudes*, NACA TN 2798, October, 1952.

[47] Hartman, A., *Orienterend Onderzoek ter Toetsing van de Cumulative Damage Hypothese* $\Sigma \frac{n}{N} = 1$ *op Klinkproefstukken*, Nationaal Luchtvaartlaboratorium, Rapport M. 1923, May, 1953.

[48] Freudenthal, A. M., *A Random Fatigue Testing Procedure and Machine*, ASTM Proceedings, Vol. 53, pp. 896–909, 1953.

[49] Marco, S. M., and Starkey, W. L., *A Concept of Fatigue Damage*, Transactions of the A.S.M.E., Vol. 76, No. 4, pp. 627–632, May, 1954.

[50] Legendre, A. M., *Tables of the Logarithms of the Complete Γ-Function to Twelve Figures*, Tract for Computers, No. IV, Cambridge, 1921.

[51] Hayashi, K., *Sieben- und mehrstellige Tafeln*, Berlin, 1926.

[52] Jahnke, E., and Emde, F., *Tables of Functions*, Leipzig and Berlin, pp. 9–21, 1938.

[53] Kennedy, A. P., *A Method for Determining the "Safe" Life of an Aircraft Wing from Fatigue Test Results*, Journal of the Royal Aeronautical Society, Vol. 58, No. 521, pp. 361–366, May, 1954.

[54] Walker, P. B., *Design Criterion for Fatigue of Wings*, Journal of the Royal Aeronautical Society, Vol. 57, No. 505, pp. 12–18, January, 1953.

[55] Searle, G. F. C., and Lindemann, F. A., *Preliminary Report on Measurements of Accelerations on Aeroplanes in Flight*, Advisory Committee for Aeronautics, Reports and Memoranda (New Series), No. 376, September, 1917.

[56] Seewald, F., *Messungen mit dem Glasritzdehnungsschreiber der Deutschen Versuchsanstalt für Luftfahrt*, Maschinenbau, Bd. 10, Nr. 23, pp. 725–727, 1931.

[57] Wills, H. A., *The Life of Aircraft Structures*, Second International Aeronautical Conference, pp. 361–389, New York, May 24–27, 1949.

[58] Jasper, N. H., *The TMB Strain Cycle Gage and Counter—An Instrument for the Statistical Determination of the Strain History of Structures*, Proceedings of the Society for Experimental Stress Analysis, Vol. X, No. 1, pp. 87–96, 1952.

[59] Foster, H. W., *A Method of Detecting Incipient Fatigue Failure*, Proceedings of the Society for Experimental Stress Analysis, Vol. IV, No. 2, pp. 25–31, 1947.

[60] Levin, M. H., *Notes on the Use of Bonded Wires to Detect Fatigue Cracks*, NACA TN 3190, Appendix A, March, 1954.

[61] Walker, P. B., *Estimation of the Fatigue Life of a Transport Aircraft*, Journal of the Royal Aeronautical Society, Vol. 57, No. 514, pp. 613–617, October, 1953.

[62] Pearson, K., *Tables of the Incomplete Γ-Function*, Cambridge, 1951.

[63] Pugsley, A. G., *A Philosophy of Aeroplane Strength Factors*, Reports and Memoranda, No. 1906, September 22, 1942.

[64] Chilver, A. H., *The Frequency of Failures within a Group of Similar Structures*, The Aeronautical Quarterly, Vol. V, Part 4, pp. 235–250, November, 1954.

APPENDIX A

Analysis of Damage Intensity and Cumulative Damage

*by Sigge Eggwertz**

The main purpose of this appendix is to obtain an explicit formula for calculating the fatigue damage caused by loads with varying amplitude, in particular gust loads, acting on an airplane structure or on a structural component. The damage is defined as a certain fraction of the damage at failure, assuming, according to the cumulative damage theory, that the damage at any stress amplitude grows linearly with the number of stress cycles applied, and that the damages of different stress amplitudes can be added to a total damage without regard to the magnitude of these amplitudes. The relation between the stress amplitudes and the corresponding number of stress cycles is approximated by simple functions both for the gust load curve and the endurance curve in the same way as in the main lecture.

Damage Intensity

The fatigue damage caused by a number of stress cycles of a certain amplitude s_a may be defined

$$D = n/N$$

If the number n of stress cycles applied is equal to the number N of cycles required for failure according to the endurance curve, the damage takes the value of unity.

In the case of gust loads on an airplane, there is no particular number n corresponding to a fixed amplitude s_a. Instead, a small interval Δs_a (Fig. A1) must be considered. If the number of gust cycles within this interval is denoted $(-\Delta H)$ the damage ΔD of the interval will be

$$\Delta D = -(\Delta H/N) \tag{A-1}$$

According to the cumulative theory which was first introduced by Palmgren in 1924,[34] the total damage in all intervals can be written

$$D = \Sigma\left(-\frac{\Delta H}{N}\right) \tag{A-2}$$

where $D = 1$ still means failure, irrespective of the order in which the different stress amplitudes are applied. It has been shown that Eq. (A-2) is not generally correct.[42, 44–49] There is some indication, however, that it may be used for gust loads without introducing substantial errors.

* Principal Scientific Officer, Structures Department, The Aeronautical Research Institute of Sweden.

Returning now to Eq. (A-1), the intensity of the damage is obtained dividing ΔD by Δs_a

$$\Delta D/\Delta s_a = -(\Delta H/\Delta s_a)(1/N) \tag{A-3}$$

If Δs_a is infinitely small the damage intensity is written

$$D' = dD/ds_a = -(dH/ds_a)(1/N) \tag{A-3a}$$

It will be shown in the following that D' is usually very small for $N \leqq 10^4$. Above this limit the endurance curve can be approximated by[29,35]

$$N = \alpha(s_a - s_{aE})^{-\beta} \tag{A-4}$$

where s_{aE} is the endurance limit (Fig. A1) and α and β are two other parameters, to be determined by test values.

The gust loads are represented by the following formula, which is in good agreement with measured values[22–27] if very high and very small gust loads are excluded

$$\log H = \log H_0 - hs_a \log e \tag{A-5}$$

where H_0 and h are two parameters.
From Eq. (A-5)

$$H = H_0 e^{-hs_a} \tag{A-5a}$$

Introducing Eqs. (A-4) and (A-5a) in Eq. (A-3a) gives

$$D' = \frac{H_0 h}{\alpha} e^{-hs_a}(s_a - s_{aE})^{\beta} \tag{A-6}$$

The maximum damage intensity is obtained by differentiating Eq. (A-6)

$$\frac{dD'}{ds_a} = -\frac{H_0 h}{\alpha} e^{-hs_a}[h(s_a - s_{aE})^{\beta} - \beta(s_a - s_{aE})^{\beta-1}]$$

$$\frac{dD'}{ds_a} = 0$$

gives the value $s_{a\,max}$ corresponding to D'_{max}

$$h(s_{a\,max} - s_{aE}) = \beta$$

$$s_{a\,max} = \beta/h + s_{aE} \tag{A-7}$$

$$D_{max}' = \frac{H_0 h}{\alpha}\left[e^{-\left(1+\frac{hs_{aE}}{\beta}\right)}\frac{\beta}{h}\right]^{\beta} \tag{A-8}$$

The corresponding value of N is obtained from Eq. (A-4)

$$N_{max} = \alpha(h/\beta)^{\beta} \tag{A-9}$$

In Fig. A2 a typical damage intensity curve has been drawn. It is important to note that the maximum is usually rather close to the endurance limit s_{aE}. For values of s_a corresponding to $N < 10^4$, where the above Eq. (A4) is no longer valid, the damage intensity is nearly always extremely small, giving no measurable contribution to the total damage. This approximation will, however, be studied analytically in the following.

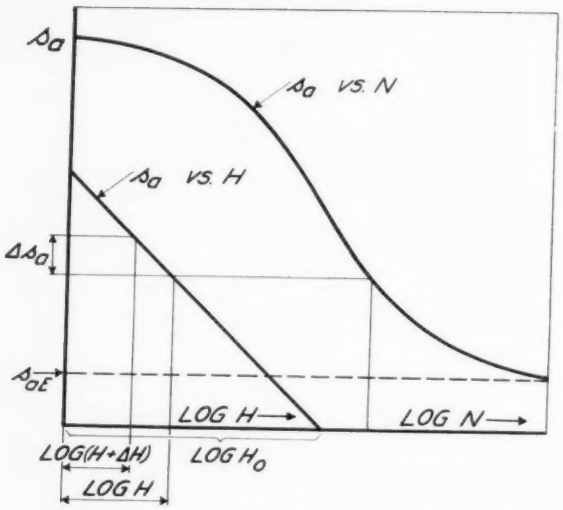


Fig. A1. Gust load curve and endurance curve.

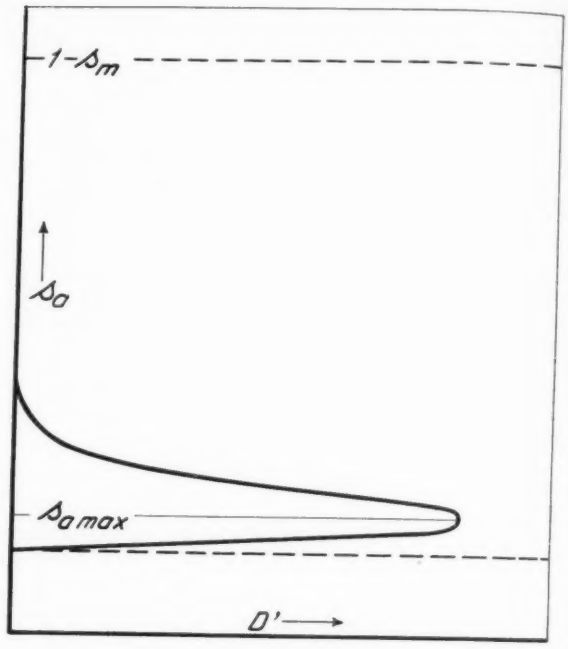


Fig. A2. Damage intensity.

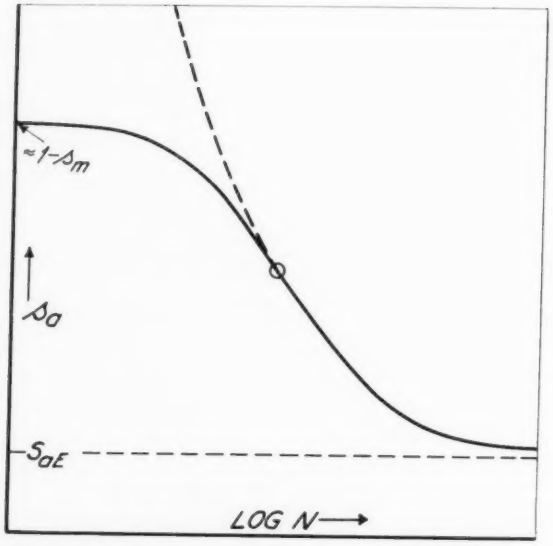


Fig. A3. Endurance curve with inflection point and extended curve corresponding to eq. (A-4) indicated.

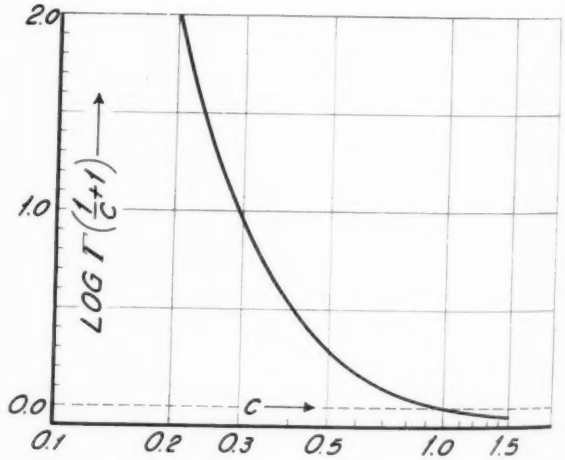


Fig. A4. Variation of logarithm of complete Gamma function $\Gamma(\frac{1}{c}+1)$

Cumulative Damage

The total damage is obtained by integrating Eq. (A-6)

$$D = \int \frac{H_0 h}{\alpha} e^{-hs_a} (s_a - s_{aE})^{\beta} ds_a$$

For values $s_a \leqq s_{aE}$ the damage is obviously zero and s_{aE} should consequently be the lower limit of the integration range. The upper limit could arbitrarily be taken somewhere between the point where the proper s_a-N curve starts to deviate from Eq. (A-4), see Fig. A3, and infinity. Using the correct endurance curve would involve a greater damage than the integration of Eq. (A-6) up to infinity. This limit will consequently give a slightly better result than the deviation point. As infinity gives the simplest formula for the damage, this limit will be chosen.

$$D = \frac{H_0 h}{\alpha} \int_{s_{aE}}^{\infty} e^{-hs_a} (s_a - s_{aE})^{\beta} ds_a \qquad \text{(A-10)}$$

Introducing

$$x = h(s_a - s_{aE})$$

$$e^{-hs_a} = e^{-x} e^{-hs_{aE}}$$

$$ds_a = dx/h$$

Eq. (A-10) is transformed into

$$D = \frac{H_0 e^{-hs_{aE}}}{\alpha h^{\beta}} \int_0^{\infty} e^{-x} x^{\beta} dx \qquad \text{(A-10a)}$$

The integral

$$\int_0^{\infty} e^{-x} x^{\beta} dx = \Gamma(\beta + 1)$$

is a Gamma-function which can be obtained from mathematical tables.[50, 51, 52] Fig. A4 shows the variation of $\log \Gamma(\beta + 1)$ with c, where $c = 1/\beta$.

The final formula is thus

$$D = \frac{H_0}{\alpha} h^{-\beta} e^{-hs_{aE}} \Gamma(\beta + 1) \qquad \text{(A-10b)}$$

In the preceding section the s_a value corresponding to maximum damage intensity has been calculated. It is now possible to obtain also a simple formula for s_a corresponding to the centroid of the area enclosed by the damage intensity curve, which might be of some interest when deciding at which stress amplitudes the fatigue tests should be run.

Taking moments about an axis $s_a = s_{aE}$ gives

$$(s_{ac} - s_{aE})D = \int_{s_{aE}}^{\infty} D'(s_a - s_{aE}) ds_a =$$

$$\frac{H_0 h}{\alpha} \int_{s_{aE}}^{\infty} e^{-hs_a} (s_a - s_{aE})^{\beta+1} ds_a =$$

$$\frac{H_0 e^{-hs_{aE}}}{\alpha h^{\beta+1}} \int_0^{\infty} e^{-x} x^{\beta+1} dx =$$

$$\frac{H_0}{\alpha} h^{-(\beta+1)} e^{-hs_{aE}} \Gamma(\beta + 2)$$

Introducing D from Eq. (A-10b) the co-ordinate of the centroid is obtained

$$s_{ac} = \frac{1}{h} \frac{\Gamma(\beta + 2)}{\Gamma(\beta + 1)} + s_{aE} = \frac{\beta + 1}{h} + s_{aE} \qquad \text{(A-11)}$$

The corresponding value of N is

$$N_c = \alpha[h/(\beta + 1)]^{\beta} \qquad \text{(A-12)}$$

Study of Influence of Complete s_a-N Curve

The upper, downward concave part of the s_a-N curve can be approximated by the equation.[35]

$$N = \alpha_1 (s_a - s_{aE})^{-\beta_1} - B \qquad \text{(A-13)}$$

where usually

$$\alpha_1 \neq \alpha \qquad \beta_1 \neq \beta$$

in comparison with Eq. (A-4).

It has been stated above that the upper part of the endurance curve does not influence the fatigue damage very much. It is necessary to know if the contribution by this part of the curve is never more than 10 or 30 per cent, but the study can be made very roughly. Consequently it seems to be permissible to introduce the simplification $\alpha_1 = \alpha$ and $\beta_1 = \beta$.

$$N = \beta(s_a - s_{aE})^{-\beta} - B \qquad \text{(A-13a)}$$

The new parameter B may be determined from a point defined by

$$N = 1 \qquad s_a = s_{a,\,N=1}$$

$$B = \alpha(s_{a,\,N=1} - s_{aE})^{-\beta} - 1 \qquad \text{(A-14)}$$

The stress amplitude $s_{a,\,N=1}$ is, of course, very close to the static failure stress minus the mean stress s_m. If all the above stresses s are supposed to be given in relation to the static failure stress, $s_{a,\,N=1}$ may thus be written

$$s_{a,\,N=1} \approx 1 - s_m \qquad \text{(A-15)}$$

According to Eq. (A-13a)

$$\frac{1}{N} = \frac{1}{\alpha(s_a - s_{aE})^{-\beta} - B} =$$

$$\frac{(s_a - s_{aE})^{\beta}}{\alpha} \frac{1}{1 - (B/\alpha)(s_a - s_{aE})^{\beta}} \qquad \text{(A-16)}$$

Introducing

$$\eta = \frac{1}{1 - (B/\alpha)(s_a - s_{aE})^{\beta}} = 1 +$$

$$(B/\alpha)(s_a - s_{aE})^{\beta} + (B^2/\alpha^2)(s_a - s_{aE})^{2\beta} + \ldots \qquad \text{(A-17)}$$

the damage intensity D' can now be written comparing Eqs. (A-3a), (A-6) and (A-16)

$$D' = \frac{H_0 h}{\alpha} \eta e^{-hs_a} (s_a - s_{aE})^{\beta} \qquad \text{(A-18)}$$

The total damage will consequently be

$$D = \frac{II_0 e^{-hs_{aE}}}{\alpha h^\beta} \int_0^{h(1-s_m-s_{aE})} \eta\, e^{-x} x^\beta dx \quad \text{(A-19)}$$

using the same x as in Eq. (A-10a).

If only the first term of η, according to Eq. (A-17), is used, Eq. (A-19) will obviously be identical with Eq. (A-10a) apart from the upper limit of the integral, which should be in this case an x-value corresponding to

$$s_a = 1 - s_m$$

The integral

$$\int_0^{x_0} e^{-x} x^\beta dx = I(u, p)\, \Gamma(\beta + 1)$$

is an incomplete Gamma-function. $I(u, p)$ has been tabulated by K. Pearson.[62] The arguments of these tables are

$$p = \beta$$
$$u = x_0/\sqrt{p+1} = h(1 - s_m - s_{aE})/\sqrt{\beta + 1}$$

Now

$$h \geqq 52 \text{ and } s_m \leqq 0.27$$

further

$$s_{aE} \leqq 0.2 \text{ and } \beta \leqq 7$$

and consequently

$$u \geqq 52 \times 0.53/\sqrt{8} = 9.7$$

A look in the tables reveals that for

$$p \leqq 7 \qquad u \geqq 9.7$$

the function $I(u, p)$ will have the limits

$$0.9999963 \leqq I(u, p) \leqq 1.0000000$$

i.e., practically

$$I(u, p) = 1$$

and thus

$$D_1 = \frac{II_0}{\alpha} h^{-\beta} e^{-hs_a} \Gamma(\beta + 1) \quad \text{(A-20)}$$

The second term of Eq. (A-17) will give the following increment in D

$$\mu_2 D_1 = \frac{H_0 hB}{\alpha^2} \int_0^{1-s_m} e^{-hs_a} (s_a - s_{aE})^{2\beta} ds_a =$$
$$\frac{H_0 B e^{-hs_a}}{\alpha^2 h^{2\beta}} \int_0^{h(1-s_m-s_{aE})} e^{-x} x^{2\beta} dx =$$
$$\frac{H_0 B}{\alpha^2 h^{2\beta}} e^{-hs_a} I(u, p) \Gamma(2\beta + 1)$$

In this case

$$p = 2\beta$$
$$u = h(1 - s_m - s_{aE})/\sqrt{2\beta + 1}$$

Assuming

$$p \leqq 14 \qquad u \geqq 7.1$$
$$0.9964 \leqq I(u, p) \leqq 1.0000$$

and consequently

$$I(u, p) \approx 1$$

$$\mu_2 D_1 = \frac{II_0 B}{\alpha^2 h^{2\beta}} e^{-hs_a} \Gamma(2\beta + 1) \quad \text{(A-21)}$$

Dividing Eq. (A-21) by Eq. (A-20) gives

$$\mu_2 = \frac{B}{\alpha h^\beta} \frac{\Gamma(2\beta + 1)}{\Gamma(\beta + 1)} \quad \text{(A-22)}$$

If B is approximated by

$$B \approx \alpha(1 - s_m - s_{aE})^{-\beta}$$

Eq. (A-22) will take the form

$$\mu_2 = [h(1 - s_m - s_{aE})]^{-\beta} \frac{\Gamma(2\beta + 1}{\Gamma(\beta + 1)}$$

Assuming

$$h \geqq 52 \text{ and } 1 - s_m - s_{aE} \geqq 1 - 0.27 - 0.20 = 0.53$$

further

$$0.8 \leqq \beta \leqq 7$$

gives the maximum of μ_2, corresponding to $\beta = 0.8$

$$\mu_{2\,max} = 0.108$$

The complete formula for D can now be written

$$D = D_1(1 + \mu_2 + \mu_3 + \ldots \mu_n + \ldots) =$$
$$\frac{II_0}{\alpha} h^{-\beta} e^{-hs_a} \Gamma(\beta + 1) \left[1 + \frac{B}{\alpha h^\beta} \frac{\Gamma(2\beta + 1)}{\Gamma(\beta + 1)} + \right.$$
$$\frac{B^2}{\alpha^2 h^{2\beta}} \frac{\Gamma(3\beta + 1)}{\Gamma(\beta + 1)} I_3(u, p) + \ldots +$$
$$\left. \frac{B^{n-1}}{\alpha^{n-1} h^{\beta(n-1)}} \frac{\Gamma(\beta n + 1)}{\Gamma(\beta + 1)} I_n(u, p) + \ldots \right] \quad \text{(A-23)}$$

With $\beta = 0.8$, which gives the slowest convergence,

$$D = D_1 (1 + 0.108 + 0.016 + 0.003 + 0.0006 + \ldots)$$

The difference between D_1, obtained from Eq. (A-20) or Eq. (A-10b), and D from Eq. (A-23) is thus always less than 15 per cent. Using values of β and s_{aE}, which are less extreme than the above, the convergence will be much more rapid and the error introduced by Eq. (A-10b) will turn out to be only a few per cent.

Discussions of the Lecture

William Littlewood, Vice-President, American Airlines, Inc.: The Eighteenth Wright Brothers Lecture, as prepared and delivered by Mr. Lundberg, has brought us a scholarly and comprehensive discussion of the fatigue life of airplane structures. I find it easy to agree with many statements of Mr. Lundberg's summary—e.g., "the dominating difficulty in the increasingly important problem of fatigue in airplane structures is the very great scatter that is typical for all its main aspects"; and "it will never be possible to predict the fatigue lifetime for an individual airplane with reasonable accuracy"; and further, "our efforts should be concerned not with the prediction of fatigue life, but with the prevention of fatal fatigue failures down to an extremely low failure rate." If I were to take any exception to these statements, it would be to stress our intention to reduce fatal fatigue failures in air transport structures to zero!

Mr. Lundberg points out that the occurrence of static structural failure appears to be of constant probability since the experience of an excessive gust velocity is equally probable at any time in the life of an airplane and that such occurrence, according to existing design standards and practices, is apparently so remotely probable as to render it a negligible risk in our safety concepts for air transport. On the other hand, he stresses the thought that fatigue damage is a cumulative process, which, after an initial period of complete freedom from vulnerability, continues to rise from a negligible probability to a high and unacceptable value later in the life of the structure.

With these concepts, I can find no disagreement. Mr. Lundberg states that his discussion is intended primarily to derive essential guides for the designer in providing some rational measures of fatigue capacity to determine the point of initial onset of serious structural fatigue probability and to provide, after design and initial fabrication, guides to check determination of what fatigue capacity has been provided. I have nothing but enthusiastic endorsement of his approach, or similar approaches, to that phase of the problem. We must, obviously, have a rational method for providing and checking the provision of adequate fatigue capacity in the airplane to be operated. Our problem remains, however, of what is adequate.

Mr. Lundberg has confined his statistical examination solely to the limited number of accidents which have occurred contributable to structural fatigue failure, whereas the much larger number of incidents which have been detected developing to varying degrees toward ultimate accidents would, in my opinion, give a much sounder basis for the determination of whether our present condition is a satisfactory criterion for future fatigue standards. According to the actually developed accident data, we apparently have a completely satisfactory fatigue safety condition, whereas examination of the incomplete information available with respect to the much more frequent incidents would raise serious doubt as to any justifiable complacency with our present condition. There is probably no area of airplane operating safety where the oft quoted comment that "aviation is not inherently dangerous but, even more than the sea, it is terribly unforgiving of any carelessness or neglect" can be truly applied than in the field of structural fatigue.

It would be presumptuous of me to discuss in detail, or to criticize, even constructively, Mr. Lundberg's masterly treatment of the highly technical phases of our problem. I suspect that I have been given an opportunity to discuss the paper much more with the idea that my deep interest and often expressed thoughts on the subject of air transport structural fatigue, as related to safety in operation, would be again reviewed with appropriate reference to that portion of Mr. Lundberg's presentation.

We are faced with two problems. One is the continued safe operation of air transport airplanes currently in use in connection with which we find varying degrees of fatigue capacity, derived more by fortuitous circumstance than by preplanned design. Secondly, we have the problem of providing in our future air transports fully adequate preplanned fatigue capacity and deriving those rules by which even such aircraft must be monitered in order to ensure freedom from fatal fatigue failure. The answers in both cases are the same—namely, inspection and repair.

Mr. Lundberg dismisses as relatively unimportant to his treatment the consideration of whether an aircraft structure be designed for fatigue with basically single elements as typified by a monospar wing or dependence on any single element of structural capacity or whether the structure be basically of a more complex multistress-path type, which may or may not be truly redundant. He does, however, point out that the probability of catastrophic failure is apparently somewhat lessened by the latter type of design.

I cannot dismiss these differences lightly, since I feel that in a full appreciation and exploitation of the latter principle lies our almost perfect protection against the occurrence of fatal fatigue failure. Mr. Lundberg stresses very properly the wide scatter of S-N values associated with materials, and the increasing scatter occasioned by variations in manufacturing techniques and inspection controls, and the grave uncertainties of the life load history of our structures subjected to major fatigue, even though we count and record the element of gust loading frequency and intensity. The tremendously wide variance in imposed stresses caused by complex combinations of stresses, even in simple elements of structure, due to air loads, temperature loads, vibration stresses, etc., still leaves us in a mass of unknowns with respect to the true life stress history of our critical parts even in any set of known operating conditions. Another unknown, and one which involves

all the possibilities of fatal failure, is our inability to determine by known inspection methods the complete absence of flaws in any single element of structure. Consequently, following the single element principle, we are always vulnerable to the fact that we have all our eggs in one basket, and, if the bottom falls out of that basket for any reason, be it unknown stresses or hidden flaws, the eggs will break. On the other hand, the inherent safety of the decreased probability of serious fatigue failure progression beyond reasonable inspection and correction in the complex structures that are fortunately common in most of our American air transport designs and many others today, and the tremendous possibilities of further reducing that probability *ad absurdum* by the intentional application of the inherently safe principles of relatively independent multistress paths, gives us full hope and expectation of satisfactory fatigue safety for the future with completely acceptable inspection and correction standards.

In order to incorporate adequately such design principles we must, of course, be able to recognize the critical design areas susceptible to disastrous fatigue failure. We can readily agree that, without subdivision, such areas include the wings, tail surfaces, and fuselage itself. There are many others, which the total design consideration must contemplate. If we are to examine, in an oversimplified consideration, the possible application of the multistress-path principle to the lower wing surfaces, we might visualize literally hundreds of parallel tension members, interrelated only as needed to provide the required shear and torsion strengths, and extending throughout the critical fatigue area. The failure in fatigue in tension of any one or even a number of these could still leave the wing with residual static strength adequate to cope with any reasonably anticipated single load condition. The incorporation in such a design of adequate *inspectability* and the further provision of ready *replaceability* would define the essential elements of this principle of fatigue safety. More by happenstance than intent, we have it to some degree in our present wing designs. We can have it to any desired degree in future designs. Our problem is to determine the structural methods best suited to make such provision wherever required in the aircraft structure with minimum weight penalty, ease of fabrication, full inspectability, and ready replaceability. This is the challenge to our fatigue research and design determination.

The inherent characteristics of this application to the problem might be summed up by stating that it incorporates (1) "fail-safe" design, where the fatigue failure of one or more elements of multistress paths still leaves adequate residual strength; (2) the controlled slow onset of developing fatigue failure by providing "impedimenta" in the paths of fatigue crack progression; (3) the arrangement of design detail so that the most highly fatigue-stressed elements shall be readily inspectable on the surface; and (4) design intended to provide ready replaceability of failed units.

If we can take a quick look at our fuselage problem, we will immediately recognize the unsuitability of contemplation of the single-element fatigue-stressed theory. The structure is inherently complex and subjected to an indeterminate combination of imposed stresses due to pressurization, wing loads, tail loads, temperature loads, vibration loads, etc., with unknown concentrations of stress under such multiple loading in the almost infinite variety of flight conditions. It seems, however, that the recognition of the same principles expressed above with emphasis on the design of our pressure vessel to arrest fatal rupture in the event of the development of an initial fatigue failure or penetration of the pressure vessel by extraneous objects will guarantee that safety that is essential. Experience has shown that simple methods can be applied to prevent the extension of initial failure or penetration of pressure vessels, including fuselages, to catastrophic rupture. We are all familiar with such methods as applied to many pressure vessels and recognize their incorporation to an unplanned extent in existing fuselages, and we can make adequate provisions in future fuselages to guarantee safety.

These are design provisions that can be developed and must be incorporated in future transport aircraft. With respect to today's operating airplanes, we are greatly dependent on inspection and repair as our primary safeguards. It seems inevitable that the ultimate course of repair requirements will lead to maintenance costs that will eventually necessitate the replacement of equipment. However, the application of rigorous inspection and corrective measures will continue to assure adequate safety, primarily, as has been said, because of the provision to a not yet satisfactory degree of the principles set forth.

So, it seems that our airplanes should be designed and checked for fatigue according to criteria such as those set forth by Mr. Lundberg, or certainly according to extremely similar principles, including the practical provision of safeguards of the types delineated herein. Thereafter, reasonable inspection and repair should ensure the almost indefinite life of our airplanes in whatever services they may be placed, and with fully acceptable safety. We are, indeed, indebted to Mr. Lundberg for his tremendous contribution to thought and progress in the essential field of fatigue life of airplane structures.

Richard V. Rhode, Assistant Director for Research, Aircraft Construction, NACA, Washington: Mr. Lundberg's paper, which he has outlined in his lecture, is too comprehensive and profound to be appraised adequately in a limited period of time. Its true significance will be known only after many designers and structural scientists have been able to study it and evaluate its potential effects on structural design, safety, and economics. Comments ventured at this time must, therefore, be limited to some rather broad questions.

To state the case briefly, Mr. Lundberg emphasizes the statistical nature of scatter and suggests that design

for fatigue be based on considerations of probability applied in such a way as to ensure a suitably low failure rate. As the data required for this process become available, he would thus elevate design for fatigue from the status of an art to the status of a science. There is much in this philosophy with which one can agree, and I do not think that the technical soundness of Mr. Lundberg's approach can successfully be challenged. At the same time, I feel that the fatigue problem is a rather urgent one and that the problem is so complex than one may properly question where research and engineering effort should be applied in order to arrive at an early practical solution.

The present fatigue failure rate of transport airplanes is much higher than the rate of one catastrophic failure in 40 years suggested by Mr. Lundberg as a suitable level of safety. In the past 6 years, there have been sufficient known cases of serious fatigue fracture throughout the world's air lines to indicate that the present rate is many times greater than this figure. Aside from the direct loss in money or lives caused by these failures, we have recently seen one transport airplane retired to limbo, its economic future wiped out, and the prestige of a nation's aviation suffer great damage as a result of fatigue failure. Thus, the fatigue problem is urgent enough, and reduction of the failure rate is obviously needed now. In seeking a solution, we must bear in mind, however, that the present failure rate is directly related to the great gains in aircraft structural efficiency which have been made over the years. There seems to be an inexorable law that says that, as structures become lighter, the fatigue life grows shorter. The great structures of antiquity are notable for their massiveness, as well as for their lasting qualities. The remarkable aircraft structures of modern times, on the other hand, are notable for their lightness and strength, but also for their short lives. The basic question, therefore, is not can we reduce the failure rate to an acceptably low and determinable level, but rather can we reduce the failure rate to an acceptable level without sacrificing the hard-won gains in structural lightness which have been made during the past 20 years.

It has not been clear to me, during my brief study of Mr. Lundberg's paper, how much weight increase might result from designing to the low failure rate he suggests. Superficially, at least, it would appear that the selection of such a low failure rate as 10^{-9} entails a degree of conservatism relative to current practice which would result in substantial weight increases of typical structures. Some idea of the amount of the increase might, perhaps, be surmised from the plots of fatigue factor, k_A, against time to failure in Figs. 30 to 34. If the present failure rate is as much as two orders of magnitude higher than that desired, these plots would seem to indicate that weight increases, at least in the tension structure, of roughly 50 per cent might be required for typical structures of limited redundancy. Perhaps Mr. Lundberg could state an opinion regarding the effect on weight, in typical cases of limited redundancy, of reduction in failure rate from the apparent current level by one, two, or three orders of magnitude.

Although less basic than the question of weight, the question of time is currently of great importance. During the past 10 years, a great deal of data have been obtained on repeated loads and on the fatigue properties of aircraft materials and structures. Also, the nature of the problem has been widely explored, and our understanding of it has greatly increased. In spite of these advances, we are still a long way from the goal of predicting either the probable fatigue life or the failure rate. As Mr. Lundberg has clearly indicated, application of the statistical method will require the acquisition of many test data, and it will be many years before these data can be accumulated, even if aircraft configuration, materials of construction, performance, and operational environment remain static. For this reason I feel that we have to find a shorter path to the immediate goal of reducing the failure rate, while at the same time avoiding retrogression in the matter of weight. This shorter path, I feel, lies in the immediate exploitation of what has variously been called redundancy, diffuse structures, multiple-load-path construction, or fail-safe design.

If we look back on the catastrophic fatigue failures that have occurred, excluding those of the pressurized fuselage of the Comet I, we find that they have occurred on single-spar or two-spar wing structures. There have also been some serious fractures and "near misses" on the slightly more redundant types, so that, by-and-large, the present unsatisfactory failure rate is attributable to failures that have occurred on the less redundant structures. This fact strongly implies, and Mr. Lundberg's paper also bears out the point, that, through application of appropriate principles of design, we "can have our cake and eat it too." That is to say, without sacrificing on weight and without increasing the repair rate over present levels, it appears that the failure rate can be drastically reduced.

The requirements for satisfactory design for fatigue can, I think, be tied to three essential principles. These principles are:

1. Slow rate of crack propagation.
2. High degree of inspectability.
3. Favorable relation between extent of fatigue crack and residual strength.

It is clear that application of these principles requires that we center our attention more on what happens after the crack starts and less on how long it takes for the crack to appear initially. The most important immediate tasks for research are, therefore, to investigate rate of crack propagation and residual strength as functions of the distribution of the material in the structure. The residual strength, as Mr. Lundberg has pointed out, must be investigated with due regard for the redistribution of load as material is lost through cracking. To be realistic, the residual strength should probably also be determined with due regard for the dynamics of both loading and fracture.

With regard to pressure fuselages, it is mainly important that the probability of castastrophic blowout be made much less than the probability of failure due to external loads or, more simply, that there be no real danger of a catastrophic blowout, even if a fatigue crack should occur. There is some indication that this goal can be attained without too much difficulty. When this is done—that is, when the design is made such that a fatigue crack can occur without causing an explosion—then fatigue of pressurized fuselages will no longer be an important hazard.

In conclusion, I should like to express my great admiration for the remarkable paper that Mr. Lundberg has prepared. Although some time will be required to assess it fully, there is no doubt that it is a major contribution on fatigue of airplane structures. In addition to the new design data and procedures that it contains, it coordinates and unifies many aspects of the fatigue problem. Understanding of the fatigue problem in itself is one of the greatest guarantors of a low failure rate. For this reason alone, Mr. Lundberg's paper should be carefully studied by all who are concerned with the structural design of aircraft.

O. J. Schaefer, Manager, Production Design, The Glenn L. Martin Company: It is with a courage born of humility that I remark on Mr. Lundberg's timely and thought-provoking lecture.

His effort toward the establishment of an overall philosophy in relation to the problem of aircraft structural fatigue is certainly well taken. Some basic planning is in order—and, in this planning, the specific responsibilities of the research institutions, regulatory agencies, air line operators, the military, and air-frame manufacturers need to be clearly understood.

The establishment of a service-life goal for the aircraft is of vital interest to all of us in any way concerned with the structural integrity of the nation's aircraft. May I suggest, in this connection, that an exact value is not so important as the determination of what has to be done to a particular airplane to make it fulfill the established service-life minimum.

Mr. Lundberg suggests that a determination be made of the proper auspices and scopes of the more urgently needed research and development programs. This appears to me as another highly important contribution. There has been a tendency, on the part of the agencies, to act independently—and perhaps to view each other's work with some scepticism. There is much wisdom, therefore, in the plan to set up research and development targets and to pool all interests concerned, as is presently being accomplished with the AIA W-76 Committee. This scheme concentrates the best talent on the problem and should save a great deal of precious time.

During the preliminary design stage, the effects of configuration changes upon the fatigue life of the aircraft are of special interest to the designer. Although considerations of fatigue may not determine the basic configuration, it is mandatory for the designer to understand the effects of configuration—as well as of environment, the load factor, and the speed of the aircraft—upon fatigue values. He must be alert to take advantage of any changes that may significantly increase the structural life of the air frame.

Detail design, as we all know, is an important factor in determining the fatigue life. Here, the designer must not only take heed of the dictates of actual experience in confirming satisfactory design criteria but must exercise great care in areas where either the material or a detail design feature may connote a sensitivity to fatigue failure.

After the release of structural drawings to the shop—drawings embodying adequate fatigue margins—back-checking of critical items still remains to be accomplished. Some of this can be done by means of repeated load tests. Revisions are in order, whenever such tests show weaknesses in terms of the original life calculations.

Another factor that bears checking is that of the assumed service history. The instrumentation of several service airplanes, to supplement existing load-magnitude and frequency information and to check any standing assumptions, will prove of unquestioned value to the designer.

Of no less importance is the matter of inspection. One of the most outstanding types of preventive maintenance is the method of "block overhaul." By dividing the entire structure into four units and, for a particular plane, by inspecting each unit at one-fourth overhaul time, it is possible, in effect, to accomplish a complete structural review at the end of each quarter of the overhaul period. Where to look will vary from type to type. It appears, however, that significant generalizations are possible, and these should help the inspector with his chores.

In closing, I wish to emphasize again the importance of our working together on this problem—particularly, of course, the air-frame manufacturers, the air-line operators, and the military. It is too early, in view of its scope, to realize the full impact of Mr. Lundberg's recommendations in the field of structural fatigue. But further study and the utilization of this material by our scientist-engineers and technicians will surely bring out its full value.

J. F. McBrearty, Chief Structures Engineer, Lockheed Aircraft Corporation: It is an honor indeed to have the opportunity to comment upon the Wright Brothers Lecture. I wish to congratulate Mr. Lundberg and The Aeronautical Research Institute of Sweden for the magnificent work that he and his organization have done in the important field of aircraft structural fatigue. The lecture that we have heard tonight is a masterful presentation of the essential elements of this entire subject.

I sometimes wonder just what is expected of a person asked to furnish prepared comment on a treatise such as Mr. Lundberg's. I suppose I could find some detail

part that I did not agree with and then belabor that point, or I could simply offer some platitudes.

However, this subject is too serious and fundamental to treat this way, so I will take the opportunity to present some of my views on this matter of fatigue life. In the first place, I submit that it is unfortunate that we continue to allude to "life" in a structure, an inanimate thing with no life at all. We scare everybody with the idea of a structure dying. Now, while a structure has no life, it does have strength, and strength is the property in which we are interested.

I feel we have all emphasized life too much. Perhaps it is a carry-over from laboratory fatigue tests of elementary coupons when in due course a specimen "dies" because of repeated loads. In a complete aircraft structure, however, the structure does not expire from a simple crack. Aircraft structures have been experiencing cracks for a long time and, while nobody likes them and we must all work hard to postpone them, I know and you know that they are inevitable—it's just a matter of time.

Now, if we reorient ourselves we find that a crack is only serious, from a safety standpoint at least, if the remaining strength is insufficient for the task. This is plenty serious. It is so serious that, in my opinion, the aircraft industry cannot afford to base its safety criterion on any analytical prediction of fatigue life even when coupled with laboratory tests of structures.

Fatigue cracks of the past have been serious when their undetected length left too little strength. Fatigue cracks of the past have not been serious when the structure remaining was still adequate for something like limit load—in other words when it "failed safe." I think this fail-safe attribute is a "must" when you consider that there are other things besides cracks that can happen—things in themselves not catastrophic such as a turbine bucket through the fuselage shell or the like—which cannot be permitted to destroy an airplane.

I feel there has been altogether too little effort on this feature both here and abroad—the feature that has prevented catastrophe in the multitudinous cases of cracks that have occurred and the feature that was absent when catastrophe did happen. I believe it is time to adopt "fail safe" as a matter of policy. What success has been achieved in the safety field so far can be attributed largely to this philosophy—multiple engines, dual controls, standby systems, alternate airports—the fundamental concept in all these other segments of the business has been that human and mechanical failure can and will occur, and the airplane system must be prepared for it.

Therefore, since I feel that we cannot (and perhaps never will be able to) *guarantee* that a crack will not form within any reasonable and practical period of time, and, even if we could there would still be the problem of other damaging events, I believe the policy of aircraft structural designers should be as follows:

1. Design the structure so that a crack, undetectible in the normal course of aircraft operation, shall leave sufficient strength for nominal limit design loads.

2. Test the structure to establish this feature by deliberately breaking typical elements whose fail-safe characteristics are not otherwise known.

3. Design the structure for allowable stresses, load spectra, and details of design in accordance with the current state of the art to avoid cracks to the extent practical and reasonable from the standpoint of the manufacturer's desire for a good reputation. There should be no regulations or guarantees on "lifetime."

This is a big order. Little is known about fail-safe structural design, but if we get busy we can find out. Most of our structures today do indeed fail safe although we may not understand why, and we may or may not have set out to achieve this feature initially. Materials have widely differing abilities to develop strength after cracking, and we must work hard to obtain this information. I think that, at long last, we may have the real virtue and the real need for ductility.

The analytical work and research on fatigue must, of course, continue. The state of the art must not be allowed to stagnate, but some of the emphasis should shift, in my opinion, from the field of crack prediction, damage theories, and such to the field of strength in damaged structures and materials. I urge The Aeronautical Research Institute of Sweden, which has done such splendid work in the fatigue field, and other research laboratories to help in this part of the business also.

I wish to congratulate Mr. Lundberg once more for his splendid work.

R. L. Schleicher, Chief Structures Engineer, North American Aviation, Inc.: Mr. Lundberg is to be congratulated on the splendid and comprehensive manner in which he has dealt with this important subject. Having recently reviewed the activities concerning fatigue in this country, my first observation is that we have many objectives and methods of approach in common. In testimony of this opinion are the many references Mr. Lundberg has made to American data and literature dealing with this subject.

Concerning a philosophy for guarding against fatigue, there are several items requiring consideration, among which are the environmental conditions and the economics of the situation. The concept of replacing members is possible only in greatly simplified designs. Most modern aircraft employ some form of shell-type construction, and hence the idea of exchanging either shear or bending material becomes economically impracticable. Instead, a graduated scale of fatigue resistance might be considered, in which it becomes more economical to replace complete components, such as control surfaces and outer wing panels at stipulated intervals, leaving the fuselage and wing center-section as relatively long-life articles. The reason for such a policy is obvious. The replaceable components are relatively inexpensive compared with the cost of system installations throughout the remainder of the airplane. The replacement period for complete components can

remain flexible, if the service life is monitored from the start by recording accelerometers or strain indicators. As an alternative, a joint program conducted by the manufacturer, the user, and a government agency could be undertaken to test a weary aircraft and the results used to ascertain the remaining fatigue life of a whole fleet.

The argument Mr. Lundberg presented concerning redundant versus simplified structures has been heard frequently in recent times. It is generally agreed that the term "multiple load path type of construction" is preferred to "redundant structure," since reference to redundant structures usually implies, to the uninitiated, that there is a great reserve of hidden strength, whereas the direct opposite might well be true.

Mr. Lundberg's key to the fatigue problem is that it must be dealt with on a statistical basis. This opinion is shared by many. As a first approach for dealing with the scatter existing in nearly all fatigue data, it is considered necessary that the data be presented in probability form as Mr. Lundberg suggests. Thus, the concept of a *P-S-N* curve will clearly show the per cent variation from the mean curve, provided, of course, that an adequate number of specimens are tested. This is a necessary first step toward improving fatigue test data and, hence, making it more useful to the designer. With increased testing facilities, the luxury of testing many specimens is slowly becoming a reality. It must be remembered, however, that there are other important variables such as concentration factors, stress amplitudes, stress ratios, etc., which must also be considered in the presentation of test data. Some of these variables may be grouped into categories to eliminate undue complicity. As an example, fatigue data on riveted and bolted joints might be listed under one level of stress concentration, whereas axially loaded members may be grouped into a second category. The stress amplitudes can be associated with the type of aircraft for which the items listed are applicable. It is trusted that in the not far distant future, fatigue test data plotted on a *P-S-N* basis can be used much the same as the 90 per cent probability values for static strength of materials now in practice.

One of the greatest practical efforts that can be made is to concentrate on design principles as a guard against fatigue. As mentioned earlier, multiple load path structures are preferred over those having single load paths. Whereas the level of static strength is not altered, the probability of failure due to fatigue (since we are accepting the statistical approach) is greatly decreased in the multiple load path structure. As a matter of design policy, North American centers its fatigue effort mainly on the testing of single load path elements to a schedule of repeated loads. Some of these fall into the category of improved design, but in every case a goal of acceptable service life is sought. Despite one's proficiency in structural design, these tests have brought to the forefront those practices that contribute to long life. Design modifications frequently lead to increased life by factors of three times or more, and this, it must be admitted, is making progress. It has become a recognized fact that fatigue is governed principally by two considerations—namely, the stress concentration factor and mean stress level. Of these, the stress concentration factor is of greatest importance and can usually be improved without serious weight penalties. Enlarged fillet radii or the displacement of a bolt pattern can frequently be accepted at no expense in weight. Other investigations in fatigue, of course, include investigations of basic materials, common fasteners, and occasionally repeated load tests of complete components. So that the designer may not be literally "carried away" in his zeal to eliminate fatigue, there is no substitute for common sense, and, as a guide, the designer should not overlook past success in similar elements of design. It is well that we not lose our sense of proportion while accepting an entirely new yardstick of acceptability in the design of extremely similar parts.

An important feature of preventing fatigue failures is the responsibility of the using agency to maintain adequate inspection, particularly as the airplane wearies in service. The designer can assist in this important phase by providing structural accessibility, and this can be provided through semipermanent doors that can be removed during major inspections. Furthermore, the airplane manufacturer can greatly assist the using agency by preparing in advance certain handbook material to act as a guide in making these inspections. It then becomes the responsibility of the using agency to use whichever means of inspection is practicable, whether it be visual, radiographic, magnetic particle, dye penetrant, or other means to detect the inception of cracks in critical areas.

In view of the serious nature of this problem and the dire consequences that are entailed, a word of caution should go forth to those whose duty it is to investigate unfortunate mishaps and accidents. There is a slight tendency among some engineers to label every failed part as having been due to fatigue. Technically speaking, of course, they are correct since even a static failure might be termed a one-cycle fatigue failure, but, as Mr. Lundberg has reminded us, a static failure is due to going beyond the yield strength of a material, whereas a fatigue failure usually takes place below the yield strength. Examination of many specimens, which failed due to repeated loadings, reveals the telltale marks of fatigue. These are discernible progressive discoloration of the fracture and minute brinelling of the crystalline structure. The extent of the discolored areas usually describes the extent of a fatigue failure prior to the last cycle. When a fracture is found in which the last fringe of fatigue damage covers the greater portion of the cross-sectional area, the probability of fatigue is enhanced. Conversely, the reverse is true. Unless some moderation in the free discussion of fatigue damage is exercised, the traveling public might well form a natural apathy toward air travel which could be quite damaging, to say the least.

Mr. Lundberg failed to mention one important aspect of fatigue which is becoming of great interest to military

aircraft designers, and that is the effect of elevated temperatures on the fatigue life of structural elements and components. A limited amount of research has been done in this field and more is urgently needed. My own knowledge includes information on a limited number of tests of single load path elements tested at elevated temperatures. The results, as usual, are widely scattered, but the trend has been that the strength at a given fatigue life reduces less rapidly than the static strength of the material, as given in the ANC-5 bulletin. This is an important part of the problem which must be dealt with more vigorously in the future.

In conclusion, I would like to say that perhaps the greatest progress that we have made to date on the subject of preventing fatigue in aircraft structures lies in the fact that *the aircraft industry has become more fatigue conscious*. The improvement in the fatigue life of elements, without regard to any particular spectrum of loads, is, in itself, something that is conducive to good design. The consciousness of more frequent and better air-frame inspection while in service is another feature that is improving. The concept of installing load or strain counters as a permanent installation in every airplane should receive serious consideration. In the final analysis, a careful assay of weary aircraft, where the service life is known through a careful service record and load counters, will enable the user to prolong the service life of his fleet, or the owner of a single airplane through a lesser series of tests to estimate the date of retirement.

L. R. Jackson and H. J. Grover, Battelle Memorial Institute: Mr. Lundberg has suggested an approach to aircraft design on the basis of a "limit" probability for fatigue failure. This approach places emphasis on the fact that there is scatter in the "fatigue" strength of supposedly identical structural elements and that the loadings imposed by gusts and maneuvers are best handled by statistical methods.

While such factors suggest that statistical analysis has a place in designing aircraft to resist fatigue, and Mr. Lundberg's closely reasoned treatment provides a framework for such design methods, we wish to suggest some additional factors that may also be worth consideration.

The basis for these suggestions is our belief that statistical methods of analysis are most useful in treating fluctuations in variables that, for one reason or another, are not amenable to control by the experimenter. By corollary, those fluctuations in variables that potentially can be controlled (although the means of control may not always be obvious prior to experiment) might well be studied by means of individual "critical" experiments designed to expose basic mechanisms.

From this viewpoint, it is instructive to attempt to separate the factors that determine the ability of an air frame to resist fatigue failure into two groups—those that are controllable and those that are not.

As mentioned above, "gust and maneuver loads" are almost entirely in the latter category. "Almost" because there is some control by the pilot in flying at slower speeds under gusty conditions so that, while the *number* of such loads is governed to a large extent by chance, the *magnitude* is partly controllable.

The familiar "scatter" in fatigue strength of smooth and simple laboratory-test pieces made from materials that are used in aircraft structures suggests that statistical methods would be appropriate here also. However, in complex structures, the situation is not so clear-cut and unamibiguous as for simple laboratory-test pieces. For example, in such a simple element of a structural component as a single-row riveted joint, the nature of the fatigue scatterband can be drastically changed merely by preceding the fatigue test by the application of a static load that "evens up" the sharing of load among the rivets. While this simple expedient may produce an impressive change in the fatigue behavior of an isolated joint, it may not be effective when the same joint is built into a complex structure, because it may not be possible to apply a static load to a structure that would equalize loads on *all* rivets without producing yielding or failure at critical locations. In this situation, it would be desirable to understand the structure well enough to know which scatterband to use in estimating service behavior. Such an understanding could conceivably be best obtained in some cases through the use of strain gages without any fatigue test at all, let alone a statistical array of fatigue tests.

Other situations in which "control" is possible include the following:

1. In laboratory tests conducted on simple structural elements, it is possible to define or, at least, to reproduce a relation between the fluctuating loads used in the test and the stresses produced in the material by these loads. In structures, the situation is not so simple. Approximate methods of stress analysis which are adequate to design for static stresses may be inadequate to predict or even to disclose components that may be important in fatigue.

2. Stress concentrations such as occur at joints, changes in section, changes in contour, cutouts, etc., seldom produce unpredictable reactions to static loading when the materials involved have the ductility usually present in metals used for air frames. However, the peak stresses resulting from such concentrations may completely dominate fatigue behavior, and it is therefore important to know the proper stress-concentration factor to be associated with each structural configuration.

As a parenthetical remark, it may be said here that the very few major fatigue failures of aircraft (known to the authors) have been attributed to local stress concentrations, and it is dubious whether these failures would have been avoided by designing on the basis of statistical allowance for scatter. Again, attention to behavior of individual tests could provide insight into the proper stress-concentration factor to use more

efficiently than might be obtained from statistical arrays.

From considerations such as these, we should like to suggest that statistical treatments such as the one suggested by Mr. Lundberg be applied only *after* all stress components resulting from any combination of loads on the aircraft are thoroughly understood at *all* critical locations and, in addition, the correct stress-concentration factor is known for every critical location in which the peak stress deviates from the average nominal stress computed by conventional stress analysis.

If these conditions are satisfied, then we believe that it may be possible to have some confidence in "limit"-probabilities criteria for design to resist fatigue failure. However, lack of knowledge of controllable factors may make probability analysis quite unrealistic.

Reply to Prepared Discussions

Mr. Littlewood: I am in full accordance with most of Mr. Littlewood's interesting thoughts, and I feel very glad about his concluding remark that airplanes should be designed and checked for fatigue according to criteria such as the one proposed in my paper. There are only a few of Mr. Littlewood's remarks I should like to comment on.

With regard to incidents versus accidents, I am fully aware that thorough investigations of incidents are at least as important as is determining the causes of accidents, be it in the field of fatigue or in any other matter of safety. However, I do not believe that it is possible, in respect of fatigue, to base the concept of safety on incidents, firstly because any definition of the "incident level" which should be used as a basis would be completely arbitrary (see Section 3.5), as an incident is an extremely vague concept and can be anything from a fatigue crack in a member of a redundant structure to a "near escape," such as a deep crack in a single-spar or two-spar wing. It would not even be appropriate to define an incident as a certain reduction of the static strength, because the seriousness of such a reduction is highly dependent on the rate of crack propagation, the degree of redundancy, and the inspection procedures applied. Furthermore, it might be pointed out that all incidents may not be reported.

The proposal that the concept of safety with regard to fatigue should be referred to fatal accidents, has not led me, however, to base the proposed magnitude of the critical failure rate on a statistical examination of accidents contributable to fatigue failure. The order of magnitude of F_{cr}, 10^{-9}, is arrived at merely on the presumption that fatal fatigue failures should be "practically unheard of" (see Section 4.2). For the U.S.A., the actual fatigue failure rate has been reasonably good, but for the whole world the failure rate since 1947 seems to have exceeded 10^{-7}, taking into account the six known catastrophes due to fatigue. In my opinion, this failure rate is completely unsatisfactory, but, on the other hand, this should not discourage us, as most of the sad occurrences in question could be considered as exceptional and due to a lack of appreciation of the fatigue problem.

With respect to the question of redundancy versus the "single element principle" I might point out that the object of my paper was not to express any specific opinions about the best principles of designing airplanes for fatigue, but rather to provide a criterion and some methods of calculation by means of which the virtues of any particular design principle, detail design, material, and inspection procedures can be evaluated, thus facilitating an efficient choice. However, I did state that it is possible to design a redundant structure so that the failure rate would fall below a required upper limit, such as 10^{-9} (see Section 3.6) and thus would approach zero. Furthermore, a comparison of Fig. 8 with Fig. 9 (or Fig. 46 with Fig. 47) brings out clearly the large difference between statically determinate and "multi-stress-path" systems.

Mr. Rhode: Mr. Rhode has clearly pointed out probably the most important practical aspects of the fatigue problem.

Mr. Rhode's plea for an early solution is fully appreciated, and I think that from this aspect the following three phases of development can be visualized, although the boundaries are vague:

(a) Safeguarding the transport planes now flying or the new planes for which the design is already frozen.

(b) Designing fatigue-safe planes within the next few years—i.e., as long as adequate design information with regard to the fatigue qualities of representative structural elements, based on sufficiently large sample size testing, is still lacking.

(c) Designing fatigue-safe planes in the more distant future when sufficient design information is available.

During the present phase (a), careful inspection and repair is undoubtedly the answer regarding planes with sufficient redundancy, whereas for non-redundant structures, such as mono-spar and some types of two-spar wings, careful inspection has to be supplemented by rather stringent requirements about replacements of main components after certain, predetermined life-times, regardless of whether detectable cracks have been found.

As regards designing new airplanes today (b), it is my belief that it is entirely feasible to apply the statistical procedure outlined in the paper without having access to fully adequate design information of the desired scope. Whenever there is a considerable gap between the intended type of design of a structural element and those laboratory specimens that are most similar to

this element and for which fatigue information is available, it should be possible to make conservative assumptions, at each stage of the statistical design procedure—e.g., with regard to standard deviation and P-case, as well as degree of redundancy and the cumulative damage theory, etc. Conducting such a statistical design analysis, in combination, of course, with a sound design based on past experience, would no doubt give a rather clear view, although on the safe side, about both the critical failure rate and the repair rate, and it would thus indicate what changes in the design, if any, must be made to fall below a required critical failure rate or a reasonable repair rate.

For future designs (c), which to an increasing extent can be based on adequate design information, the statistical fatigue analysis can be conducted with correspondingly greater accuracy.

As to the questions whether we can "reduce the failure rate to an acceptable level without sacrificing the hard-won gains in structural lightness" and specifically, "the effect on weight, in typical cases of limited redundancy, of reduction in failure rate from the apparent current level, by one, two, or three orders of magnitude," I wish to say the following: By the last question, Mr. Rhode indicates his opinion that an increase of weight might be confined mainly to cases of limited redundancy. This, I think, is true and is an important observation. Although Figs. 30–32 are based on rather insufficient sample sizes, they can give a rough indication about the weight increase for a non-redundant wing structure, if it is assumed that the probably somewhat too large scatter band arrived at by applying such a high confidence coefficient as 0.95 roughly outweighs the uncertainties due to the difference between these laboratory tests and an actual design. The figures indicate that a decrease of the failure rate by a factor of 10 would necessitate an increase of the fatigue factor, k_A, of the order of 7 to 10 per cent at a limit life of around 50,000 hours, which would imply an increase in area of 15 to 20 per cent for a decrease of the failure rate from 10^{-7} to 10^{-9}. It must be borne in mind, however, that such an increase of the area is in most cases limited to a few fatigue-sensitive joints with comparatively short lengths, which means that an area increase of 20 per cent might not imply a greater increase of the weight of the wing structure than around 1 or 2 per cent.

In the case of redundant, or "multiple-load-path," structures, it is still more difficult to make a general statement about the necessary increase in weight. For highly redundant structures (4 spars or more) and for pressure cabins, thoroughly designed as "fail-safe" structures, it might often be possible that k_A equal to 1.0 for most elements would be quite sufficient to ascertain a failure rate of for instance 10^{-9}, but then a tolerable repair rate might necessitate considerably higher k_A-values and thus a critical failure rate much less than 10^{-9}.

For intermediate cases of "moderate redundancy"—for instance, some types of three-spar wings—either the failure rate criterion or a demand on a maximum repair rate might determine the k_A-values—i.e., the structure weight. It is of course not possible to state anything about weight increase compared with current designs, except that such increases, if they have to be made at all, will be very small. In view of this and what is said above, I do not believe that the weight question is an argument against the proposed failure rate criterion and statistical analysis. On the contrary, it is felt that the statistical method is a tool for determining the most efficient way of distributing the weight.

I agree completely with Mr. Rhode regarding the importance of extended research on rate of crack propagation and the relation between extent of fatigue crack and residual strength. The latter item I should like to extend to cover also the residual strength of, for instance, a multi-spar wing when one of the spars has failed completely. The remaining strength necessary to classify the structure as fully "fail safe" has, of course, to be determined by a statistical treatment, considering the probability of encountering a strong, destructive gust during the same flight in which the spar failed.

Finally, I might sum up my reply to Mr. Rhode's interesting comments by expressing the opinion that his recommendations for satisfactory design for fatigue are not to be regarded as an alternative "shorter path" to ascertaining sufficient safety; they can all be considered and treated as sound design principles, the virtues of which will appear clearly by a statistical analysis.

Mr. Schaefer: I quite agree with all the points of view expressed by Mr. Schaefer.

Mr. McBrearty: I very much appreciate Mr. McBrearty's statement that "the aircraft industry cannot afford to base its safety criterion on any analytical prediction of fatigue life," as has also been emphasized in the paper. In order to avoid confusion, it might be pointed out, however, that attempts to predict fatigue life are one thing, and an impossible task, and a statistical calculation of the fatigue life referred to, or tied up with, a certain probability of failure is quite another matter, an altogether feasible one. The latter method is, in fact, nothing but a rational guide for preventing structural failures, caused by fatigue, and has only the word "life" in common with the "prediction approach." However, fatigue cannot, of course, be dealt with without accepting its most important feature, life.

I also believe that Mr. McBrearty's emphatic stressing of the "fail-safe" concept is highly appropriate. However, as stated above, a fail-safe design is not an alternative to the statistical criterion and analysis proposed; it is rather a rational method of complying with, consciously or subconsciously, a certain maximum failure rate.

Regarding the question of whether we should have any regulations, I cannot agree with Mr. McBrearty's negative attitude on this point. In view of all the

tragic fatigue failures that have occurred, I can see no reason whatever why this particular field of aeronautics should not be covered by an appropriate criterion or requirement. Briefly, there are two main reasons for having such a fatigue requirement:

(1) A recognized criterion of the type suggested, combined with statistical analysis, would be of considerable value for the designer, as it would enable him to work to a clearly defined level of safety and to comply with it with the least possible structure weight.

(2) Without a numerically specified rule, some designers in some countries, where the aircraft industry might be less advanced, may not appreciate the seriousness of the fatigue problem, and, if they do appreciate it to some extent, they might feel that a design, the fatigue qualities of which correspond to a failure rate as high as say, 10^{-6}, complies with the current "requirement," which prescribes that the probability of fatigue failure should be "extremely remote." This could easily lead to numerous fatal failures and narrow escapes. It might, of course, be stated that such sad occurrences would be of no consequence for the well-reputed aircraft industries and airlines of the big aviation countries, but such an opinion, I feel, would be dangerous and inconsistent with the basic idea underlying ICAO—namely, that of attaining a uniform level of safety throughout the world. This idea is based on the self-evident fact that any accident in commercial aviation in any part of the world, regardless of operator or type of airplane, does reduce confidence in aviation.

It might be added that, although the requirements are not in the first place written for the highly experienced and reliable manufacturers and operators, it should be recognized that it is obviously in the interest of these well-established companies, both with regard to competition and the public's confidence in air transportation, that international requirements, worthy of the name, are issued and applied by all competitors. It would appear to be easy for those designers, who advocate fully "fail-safe" designs, implying a failure rate normally much less than 10^{-9}, to agree wholeheartedly with a minimum requirement of the proposed type.

Mr. Schleicher: Mr. Schleicher has presented a great number of interesting and important viewpoints. I agree with most of them and will therefore only make a few comments.

I was pleased to hear Mr. Schleicher's opinion that the idea of exchanging shear and bending material is "economically impracticable" in shell-type constructions. Such replacements are, however, rather common in the normal maintenance work of many operators. For future airplanes, it seems more rational to design various components for different service or limit lives, as Mr. Schleicher suggests. Such items as the fuselage and wing center section would then be designed for the limit life of the "airplane," and repairs of these items would be of a minor character and rather exceptional. Other components, such as the outer wings, would be designed for a shorter limit life, for instance, one third of that of the major portions of the airplane. The determination of the fractional life of such components is a straight-forward optimum problem, in which the gain in payload corresponding to a reduced weight (small k_A) due to a shorter limit life has to be balanced against the increased cost of the more frequent replacements of such components. Such a policy is entirely consistent with the failure rate criterion and the statistical analysis outlined in the paper; it should merely be observed that the sum of the critical failure rates of all the components of the airplane should not exceed the total maximum failure rate.

Regarding the possibility of using fatigue test data plotted on a P-S-N basis in about the same way as the 90 per cent probability values for static strength of materials, it might be said that these two topics are of a rather different nature. In case of static strength, the 10 per cent deficiency is efficiently covered by the static factor of safety, whereas designing to a certain P-level in respect of fatigue—for instance, to $P = 1$ or 0.1 per cent—does not seem equally rational, in the first place because one and the same P-level would give widely different failures rates (= level of safety) for different service lives, as well as for various degrees of redundancy.

The reason I have omitted the effect of elevated temperature on the fatigue life, which is to-day of such great interest to military aircraft designers, is that my paper was confined to transport airplanes, in the first place commercial ones. I might have mentioned, though, that within a few decades kinetic heating might constitute an additional problem also for transport airplanes, and already to-day elevated temperatures in the neighbourhood of engines are a problem. It might be emphasized that whenever elevated temperatures impair the fatigue life or introduce life-related effects, such as creep, it would be logical and feasible to include all such life-related conditions in the required critical failure rate.

Messrs. Jackson and Grover: Messrs. Jackson and Grover emphasize the fact that there is often a very substantial difference in the first place between the loading and stress conditions in laboratory testing of elements and the actual conditions in a complete airframe. This fact, which has been extensively discussed in the paper, leads the discussors to separate the factors that determine the ability of an airframe to resist fatigue failure into two groups: those that are controllable and those that are not. It is, furthermore, stated, if I have correctly understood the comment, that although the latter factors, the uncontrollable ones, in principle can be treated by statistics, a statistical analysis of the complete airframe would be impossible or of little value as long as the "controllable" factors are unknown.

This whole question is undoubtedly a very vital one, the answer of which is of importance for the feasibility

of the failure rate criterion and statistical methods for compliance with it.

Messrs. Jackson and Grover express the opinion that in a number of cases the "controllable" factors can hardly be determined, which, as I see it, would imply that they are not "controllable" at all. I feel that a greater clarity could be obtained by calling uncontrollable factors "factors with statistical scatter" and by calling controllable factors "factors with other uncertainties" or "matters of ignorance," as has been done in the paper.

Factors with statistical scatter or, for shortness, "Type I," would be defined as factors, the uncertainty of which is mainly of an inherent nature and can be expressed by statistical parameters, such as mean value and standard deviation provided that test results from large sample sizes or corresponding information are available or can be assumed with a reasonable confidence. Factors with other uncertainties, "Type II," would be defined as factors, the uncertainty of which cannot be expressed by statistical parameters, at least not at a certain level of knowledge.

In the comment the "gust and manoeuvre loads" are "almost entirely" classified as a factor with statistical scatter (Type I). I quite agree, but I have not stated that the applied loads should best be handled by statistical methods. On the contrary, I feel (see for instance Section 3.3) that recording as closely as possible the load history of each individual airplane (or the accumulated damage effect of the applied loads), in the first place by fatigue metering, constitutes an exceptional and good method of avoiding the lion's share of probably the greatest uncertainty of all—i.e., where and how the individual airplane is flown.

The most important factor that in principle belongs to Type I seems therefore to be the fatigue qualities of the elements of an airframe, because these qualities by nature have a statistical scatter. The accuracy with which this scatter can be determined in terms of statistical parameters is, however, dependent on available design information, as was discussed for instance in Section 6.1. If there is a considerable gap between the actual elements built into the structure and the specimens of the laboratory tests which are closest to the actual elements, this gap constitutes a factor of Type II. The most rational way of dealing with this particular "gap" in knowledge is to make a conservative assumption regarding the difference in the statistical parameters between the laboratory tests and the actual elements.

The most important factor of Type II, the uncertainty of which cannot be grasped, it seems, by statistical methods, is, as the discussors have emphasized, the actual distribution of the loads and stresses in a complex structure—for instance, a redundant one. This is a typical "matter of ignorance" and the magnitude and significance of this ignorance is, of course, mainly dependent on the accuracy with which the interior static load and stress distribution is determined under given conditions of the applied loads. It seems evident, although a bit paradoxical, that designing for fatigue calls for a more accurate and thorough static analysis, by means of theoretical calculations and tests using strain gages, in particular below the limit load, than is necessary for determining the ultimate static strength. Such a more extensive study of the load distributions is an entirely feasible thing and should be conducted regardless of the method used for designing for fatigue. In some cases it might be possible to determine the interior load distribution with a very high degree of accuracy, implying that this uncertainty is eliminated and that the loads can thus be entered into the statistical analysis as known quantities. In most cases, there might remain a smaller or greater uncertainty, and then, again, conservative assumptions have to be made regarding their magnitude; but, even so, the statistical analysis, as outlined in Section 8.4, is entirely possible and rational. I, therefore, cannot subscribe to the opinion that "lack of knowledge of controllable factors may make probability analysis quite unrealistic." On the contrary, I believe that the true significance, in respect of the fatigue qualities of a structure, of such uncertainties can best be evaluated by a statistical analysis of the structure.

With regard to the influence of stress concentrations, it might only be said that this has hardly anything to do with the methods used for designing for fatigue. With the statistical method it is, of course, equally important as with any other approach to reduce the stress concentrations as much as possible and to study and assess their effect, including their influence on scatter, by individual tests with as large sample sizes as possible. It might be added that the statistical design procedure outlined in the paper is something much more than a "statistical allowance" for scatter. It is a method of consciously designing for compliance with a specified level of safety, taking also such important points as the degree of redundancy and the effectiveness of inspections into account.

In addition it might be pointed out that "check testing" (Topic B), in particular of complete components manufactured by normal production techniques, is an excellent supplement to designing for fatigue by the statistical method. Besides the possibility that such tests might disclose as fatigue-sensitive spots or parts of the component which were not suspected as fatigue-weak in the statistical analysis (and then this analysis has to be modified), such tests serve the important purpose of giving an overall check of the accuracy of the statistical analysis at the design stage, in particular, whether the conservative assumptions of the type mentioned above were reasonable. It might, furthermore, be emphasized that, while a few check-tests, due to the scatter, can reveal very little about the "safe" or limit fatigue life of the structure (or about the repair rate to be expected), if studied as an isolated test-series, they do tell us a great deal more if they are evaluated in the light of a statistical analysis of the same structure. Only in the latter case they can serve as a reliable guide for the determination of a safe fatigue life compatible with a required level of safety.

I should finally like to express my appreciation to the discussors for their interesting comments on the paper.

Separation of a Supersonic Turbulent Boundary Layer*

S. M. BOGDONOFF† AND C. E. KEPLER‡
Princeton University

SUMMARY

The separation of a supersonic turbulent boundary layer under the effect of a strong adverse gradient has been studied at $M \sim 3$ by examining in detail the phenomena of flow over a step and shock-wave boundary-layer interaction. Wall static pressures, total head surveys, and optical techniques, including the use of color Schlieren, were used to provide a model of the separation phenomenon. It was found that: (1) When the separated region is small, the phenomenon appears to be different from when the separated region is large. For small regions the gradients are stronger, and large changes can occur for small changes in the disturbance. (2) Separation at $M \sim 3$ occurred at a pressure ratio of about 2, about two to three boundary-layer thicknesses from the start of the interaction and was unaffected by the flow downstream. (3) A considerable pressure rise occurs *after* the separation point. The peak pressure ratio of about 2.6 occurs approximately six to eight boundary-layer thicknesses downstream of the start of the interaction. (4) These results, combined with other investigations, indicate an extremely small change of separation pressure ratio with Mach Number. (5) The effect of Reynolds Number on the phenomenon appears to be negligible. (6) The detailed model, showing considerable pressure rise after separation, appears to be susceptible to the type of theoretical treatment proposed by Crocco and Lees. However, more results on mixing rates are needed for its direct application.

NOMENCLATURE

- x = distance measured along tunnel wall from face of step, in.
- x = distance from point of impingement of shock if no boundary layers were present, in.
- y = distance measured perpendicular to the wall, in.
- h = step height, in.
- P = local static pressure
- P_0 = stagnation pressure
- P_1 = free-stream static pressure
- P_2 = static pressure behind generated shocks (as measured on shock generators)
- P_t = total head pressure
- M = Mach Number
- δ = boundary-layer thickness, in.
- θ = nominal wedge angle (flow deflection angle, degrees)
- V = velocity, ft. per sec.

Presented at the Aerodynamics Session, Twenty-Second Annual Meeting, IAS, New York, January 25-29, 1954. Revised and received December 6, 1954.

* This work has been carried out under the joint sponsorship of the Office of Naval Research, Mechanics Branch, and the Office of Scientific Research, United States Air Force, Contract N6-onr-270.

† Associate Professor, Aeronautical Engineering Department, Head of the Gas Dynamics Laboratory.

‡ Formerly, Research Assistant, Aeronautical Engineering Department. Now, with the Research Division, United Aircraft Corporation.

INTRODUCTION

IN HIGH-SPEED AERODYNAMICS, a major deviation of the actual flows from those predicted by potential theory are caused by viscous or boundary-layer effects. The deviations are particularly strong in adverse pressure gradients. The high-speed viscous effects cannot be adequately determined by theoretical methods, and, in fact, there are still many unanswered questions about boundary layers in low speed flows. This is especially true for turbulent boundary layers. A large number of experimental tests and the quasi-theoretical work provide information as to general trends for flows over flat plates or in mild gradients. However, there is little available information to predict separation or the flows once separation occurs. In supersonic flows, these phenomena are somewhat more critical since there are many regions of extremely strong gradients—i.e., in diffusers, on control surfaces, and at the trailing edge of wings. It is difficult to study boundary layers in supersonic flows without an understanding of the subsonic phenomenon, even though in some ways the supersonic problem is simpler. The effects of the pressure gradient can be limited in extent, separated regions can be buried in completely supersonic fields, and, in general, more control can be exerted in the imposition of desired gradients.

In the last few years, many studies of supersonic boundary layers in adverse gradients have been carried out both theoretically and experimentally.[1-21] It is already well known that laminar layers separate for extremely small imposed gradients, and there seems to be some indication that the theoretical work available roughly predicts the phenomenon. The results for turbulent boundary layers, which can support considerably higher pressure gradients, are of more practical interest and have been the subject of rather intensive recent work.

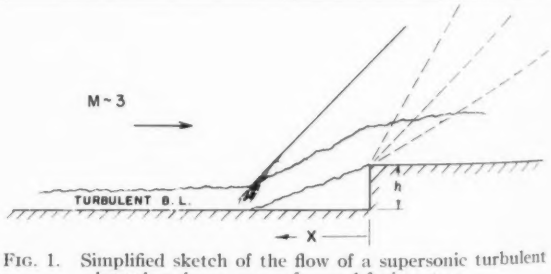


FIG. 1. Simplified sketch of the flow of a supersonic turbulent boundary layer over a forward facing step.

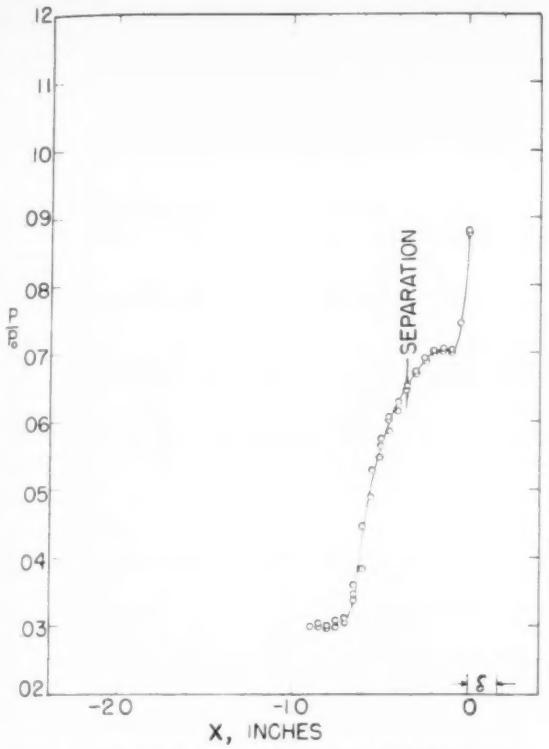


FIG. 2. Wall static pressure distribution for 0.10-in. step.

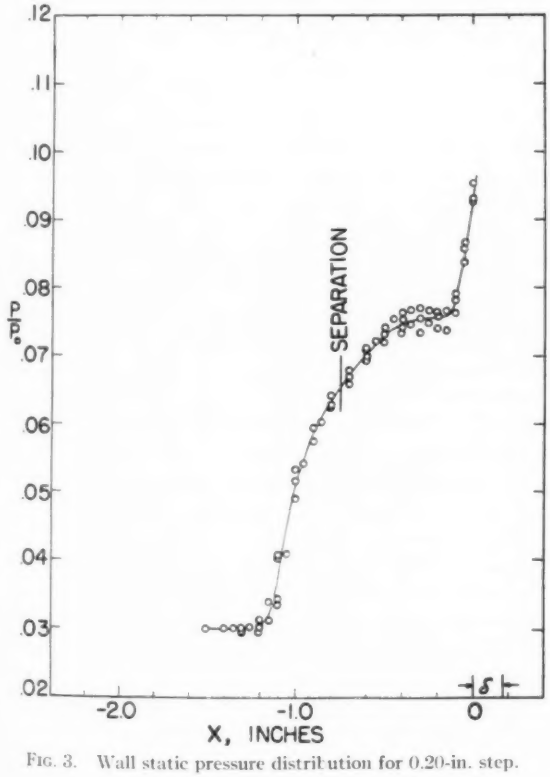


FIG. 3. Wall static pressure distribution for 0.20-in. step.

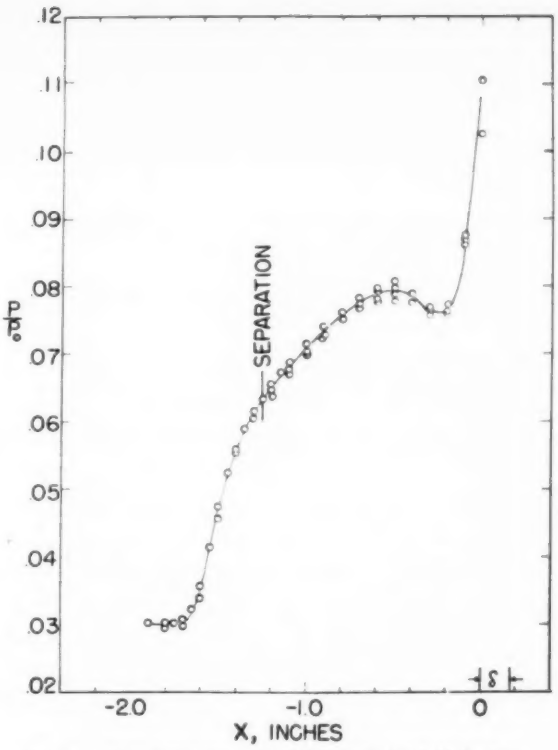


FIG. 4. Wall static pressure distribution for 0.30-in. step.

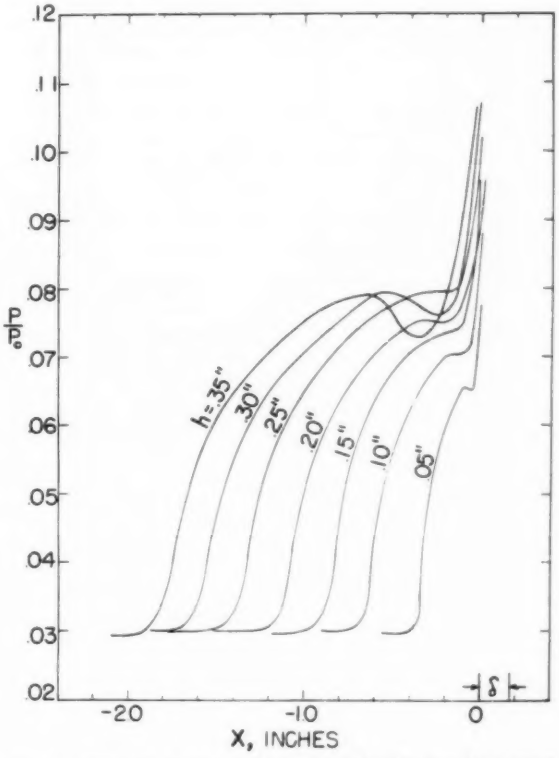


FIG. 5. Summary of wall static pressure distributions for a range of step heights.

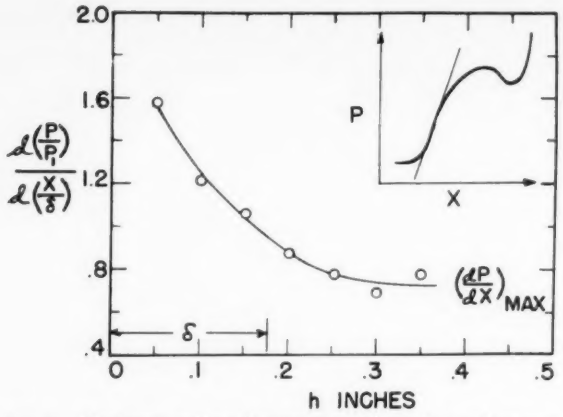


FIG. 6. Maximum wall static pressure gradients for various step heights.

This paper is particularly concerned with the separation of a turbulent boundary layer for the special case where separation is caused by extremely strong gradients—for example, an incident shock, a corner, a sudden change in flow area, or transition from supersonic to subsonic flow through a normal shock. It is important to define separation since it has not been done in the case of most reported investigations. In addition, some method must be available for determining the actual separation point. The usual definition is the point of zero skin friction or zero velocity gradient at the wall. Direct skin friction measurements have been made for cases of zero pressure gradient but are not satisfactory in the presence of the high gradients observed in these tests. The present study has, therefore, defined the separation point on the basis of zero velocity gradient normal to the wall. A small total pressure probe that permitted readings approximately 0.004 in. from the wall (total boundary-layer thickness of 0.17 in.) and the measured wall static pressures were used for the determination of velocity gradient. The same method was used for all tests, and the results are, therefore, consistent within themselves.

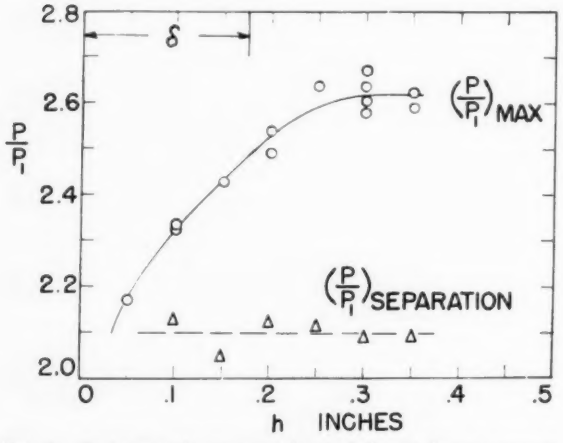


FIG. 7. Pressure ratio at separation and the maximum pressure ratio measured for various step heights.

The information that is needed is (a) the strength of the disturbance which will separate a turbulent boundary layer, (b) the effect of different types of disturbances, (c) the effect of Mach and Reynolds Numbers on the phenomenon, and (d) the flow pattern downstream of the separation point and how is it influenced by (a), (b), and (c). More fundamentally, what is really needed is (e) a detailed model of the separation phenomenon and the mechanism of separation.

The present paper is part of a detailed study of separated flows. It presents some results obtained thus far and an attempt to construct a model of the separation phenomenon. Together with other available data, it presents a picture of the phenomenon as a guide for further theoretical and experimental study. Some of the results are perhaps immediately applicable to current problems.

Experimental Studies

Experimental Difficulties

The two-dimensional phenomenon appears to be the simplest case to work with from a theoretical point of view. It is, however, unlikely that many two-dimensional separation phenomena would occur in practice, although there are some cases that approach it closely. If an experimental study is to be made with the primary intent to provide fundamental data, the two-dimensional approach seems practical. However, since the separation of a supersonic boundary layer is going to be associated with strong gradients and mixed flows, it is extremely important that the assumption of two dimensionality be examined critically. Results reported in reference 9 and some recent unpublished work of the Gas Dynamics Laboratory, Forrestal Research Center, Princeton University, have shown that a considerable number of tests, supposedly two-dimensional, actually are far from it. The effects of using shock generators, wedges, or steps which do not completely span the tunnel give results that deviate considerably from those obtained when the flows are proved two-dimensional. For example, the final pressure that one would expect to get for flow in a corner must approach the theoretical calculated pressure obtained by considering a shock at the junction of the two walls. If the flow is not two-dimensional, however, the final pressure may be as much as 50 per cent less than the theoretical one. The theoretical value is approached closely by a completely two-dimensional phenomenon. The three-dimensional effects make it extremely difficult to use directly a large amount of the available published data. Additional results are also questionable because of extraneous effects of expansions striking the interaction region. These expansions that come from corners and the rear of wedges can have large effects on the interaction and separation phenomenon.

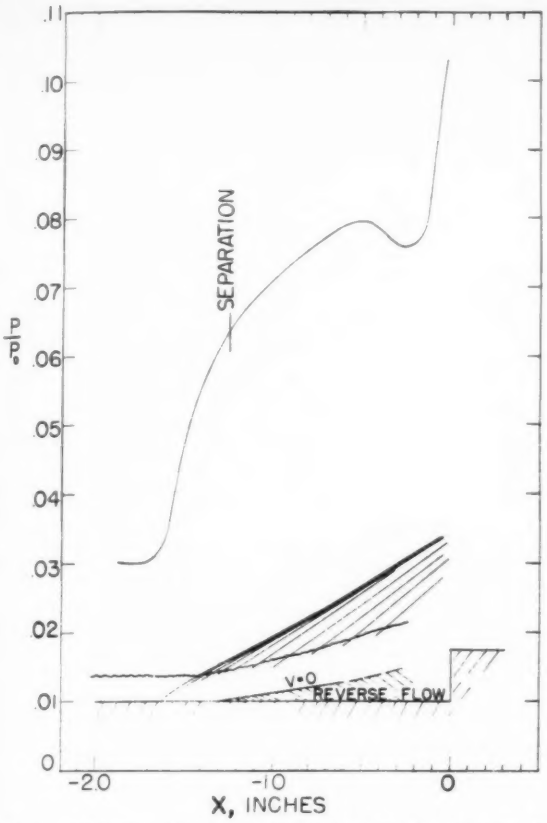


FIG. 8. Schematic drawing of interaction for 0.30-in. step in juxtaposition with the wall static pressure distribution.

Step Studies

One technique for studying separation has been to force separation by means of an upstream facing step mounted on a flat surface (Fig. 1). If tests are run at single Mach Number and with a given boundary layer, the only parameter that can be varied is the step height. Such a series of tests have been made and reported in reference 14. The experimental setup is relatively simple. The technique used was to move the step with respect to a single orifice in the wall, thus obtaining data on wall static pressure at stations as close together as desired. Static pressures were also obtained on the face of the step to obtain additional information on the gradients in this region. An extremely small total head tube (0.007 in. in height as compared to the 0.17 in. initial boundary-layer thickness) was used to survey in the separated region and was pointed upstream or downstream as desired.

Wall Static Pressure Results

Typical wall static pressure curves for several step heights are shown in Figs. 2 to 4. The separation points are indicated on each of these curves. A summary of wall static pressure results for all the steps tested are shown in Fig. 5. As is obvious, there is a change in these curves as the step height varies. Analyses of the above curves are shown in Figs. 6 and 7 and may be summarized as follows: Separation occurs at a pressure ratio of about 2.1 independently of the step height. The peak pressure ratio depends on the height of the step up to about two boundary-layer thicknesses. Above this step height, the peak pressure ratio is constant. The pressure gradient is a maximum for small step heights and becomes constant at a step height of approximately two boundary-layer thicknesses. This appears to be opposite to physical reasoning, since one would expect a small step to affect the lower part of the boundary layer with its associated lower energy level. The pressure gradient results for small steps have a parallel in the weak shock boundary-layer interaction discussed later in the paper. The occurrence of separation causes no strong change in the general phenomenon. The peak pressure rise that is experienced is the result of the pressure rise up to the separation point plus an additional rise due to the mixing phenomenon. The mixing phenomenon may account for as much as 20 per cent of the overall pressure rise. There appears to be a definite limitation to the pressure caused by this phenomenon, since increasing step height does not give continuously increasing peak pressures.

Total Head Surveys

Since the interaction region seems to reach some sort of consistency with step height for large step heights,

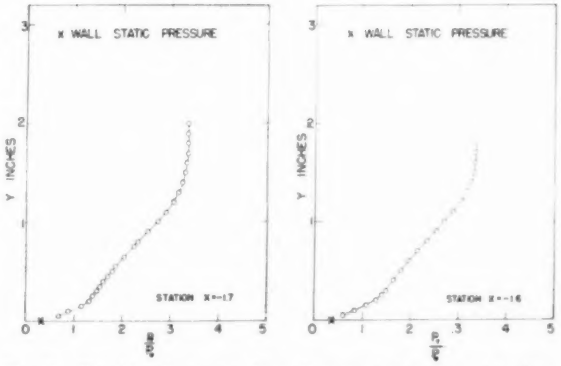


FIG. 9. Total head surveys through separated regions at various stations for 0.30-in. step.

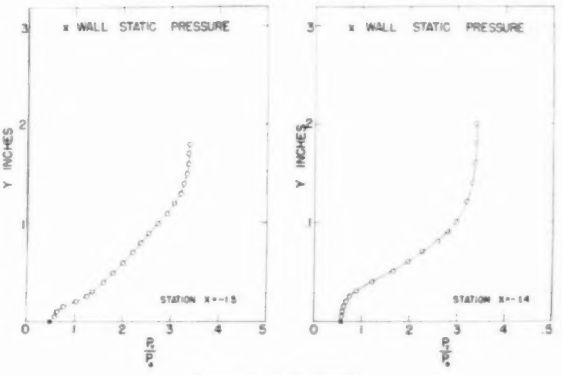


FIG. 9. Continued.

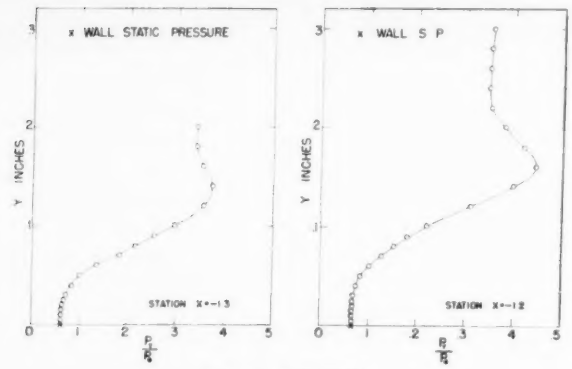


FIG. 9. Continued.

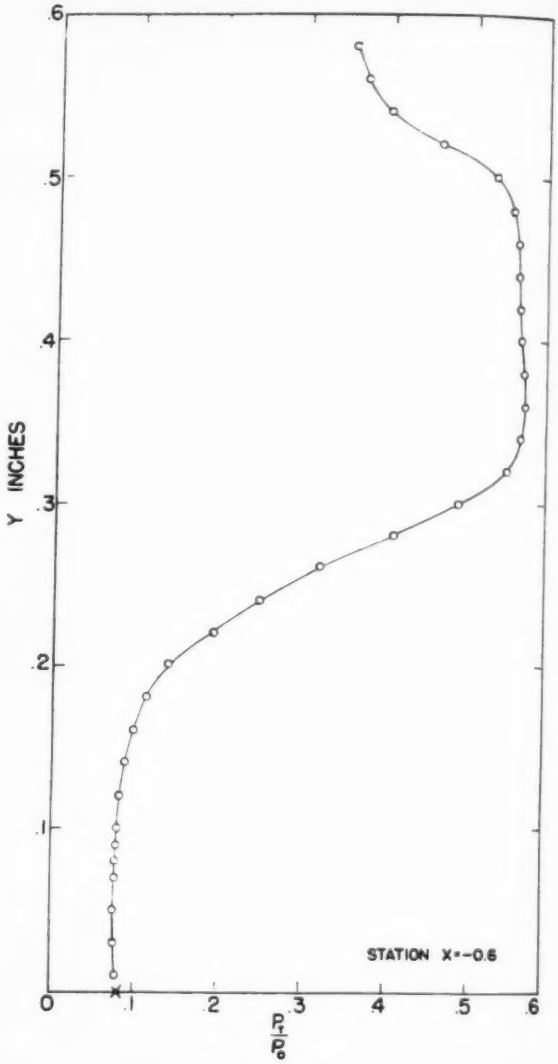


FIG. 9. Continued.

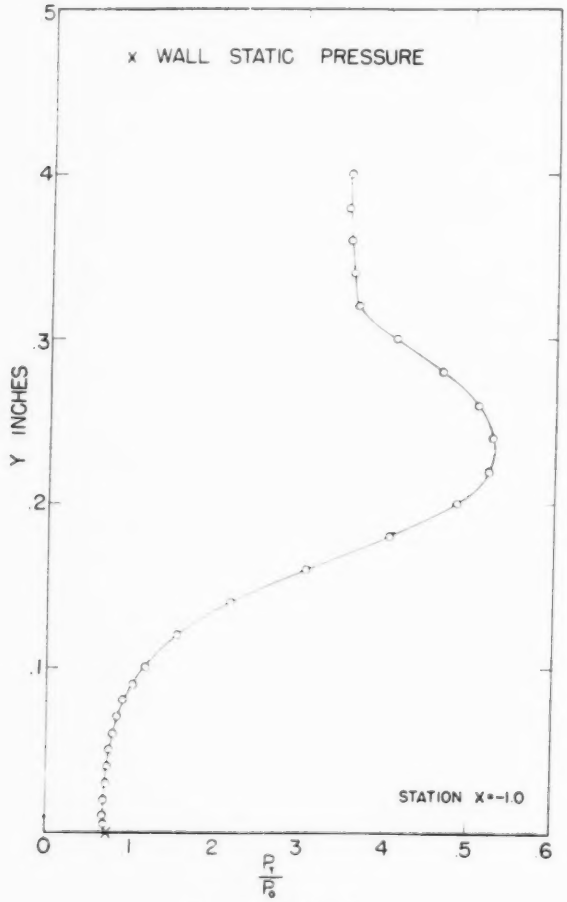


FIG. 9. Continued.

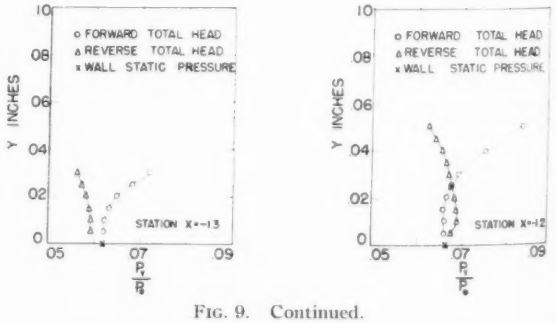


FIG. 9. Continued.

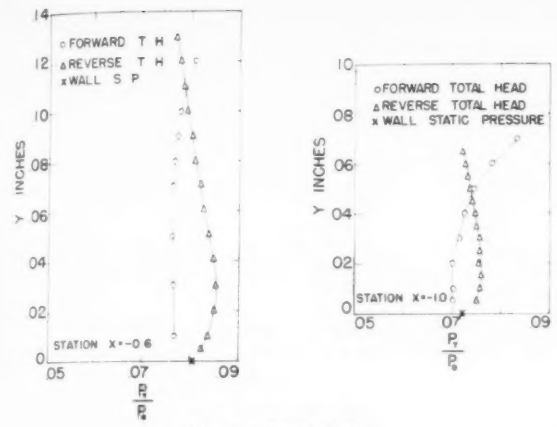


FIG. 9. Concluded.

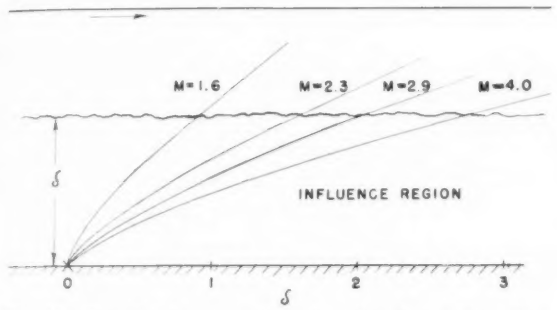


FIG. 10. Calculated propagation of a disturbance through a turbulent boundary layer assuming a 1/7 profile.

detailed total head surveys were made through the phenomenon for a 0.30-in. step (Fig. 8). The total head tube mentioned previously was arranged to face upstream as well as downstream. By making surveys at each station with both positions of the tube, the flow direction at each station could be obtained. These small-scale studies carried out near the wall are shown together with the full profiles in Fig. 9. At station -1.7 the interaction has just started and the boundary layer is undisturbed. At station -1.5 (over a boundary-layer thickness downstream of the beginning of the interaction) the wall pressure has risen to one and one-half times the initial pressure and the lower part of the boundary layer has begun to show a deceleration. Station -1.4 shows still no effect of the phenomenon on the outer edge of the boundary layer. At station -1.3, a small compression fan has reached the outer edge of the boundary layer, but the detailed study with the upstream and downstream facing tube shows that separation has not yet occurred. The pressure at this point is about twice the initial pressure. At the next survey station, -1.2, separation is indicated by the detailed survey. Further downstream at stations -0.8 and -0.6, the separation region has become large, there is a wide mixing region, a relatively uniform external region, and then the shock caused by the coalescing of the compression waves from the beginning of the interaction.

The above results are sufficiently detailed to construct a model of the separation phenomenon. It is instructive before doing so to examine how a disturbance on the wall is propagated through a boundary layer. A typical turbulent boundary layer in supersonic flow can be approximated by the well-known 1/7 power profile. A disturbance on the wall will propagate outward into the stream from the sonic line in the boundary layer along a Mach line. Since the Mach Number through the boundary layer varies, this Mach line will be curved and will take the approximate shape shown in Fig. 10. These calculations have been carried out for several Mach Numbers to show the effect of this variable. This figure gives an indication of the region of influence of a small disturbance such as must occur at the beginning of the separation phenomenon. Using the results for Mach Number 2.9 and combining with the previous detailed surveys, the model shown in Fig. 11 is obtained. Several important points should be noticed.

(1) For the initial region covering approximately two boundary-layer thicknesses, there is no separation even though the pressure rises to almost twice the initial value on the wall. In this region, normal gradients are extremely large, about twice that along the wall. The boundary layer in this region experiences considerably higher pressures near the wall than at the outer edge, a condition that invalidates the use of the ordinary boundary-layer equations in this area. The flow is completely shock free in this region (to within the accuracy of the experiments).

(2) At just about the station at which separation occurs, enough of the compression waves originating at the front of the interaction have coalesced to form an extremely weak shock just outside the boundary layer. This shock gets stronger as it extends further

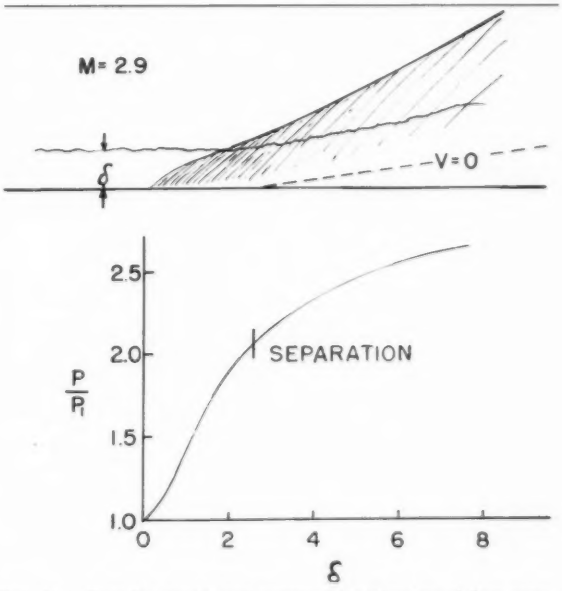


FIG. 11. Detailed model and pressure distribution of flow separation ahead of a step.

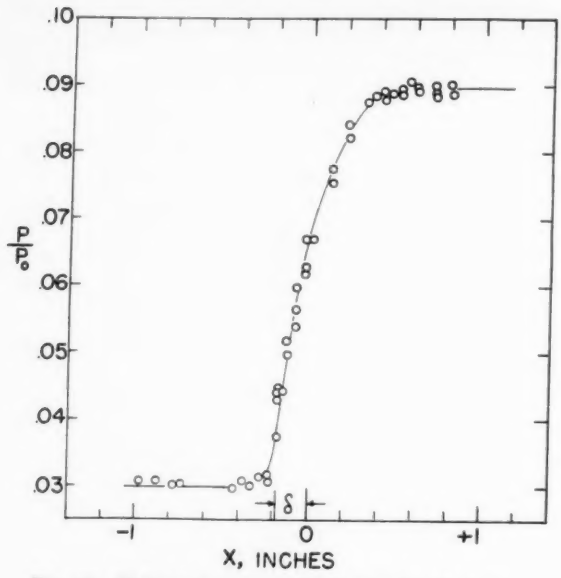


FIG. 12. Wall static pressure distribution for 8° shock.

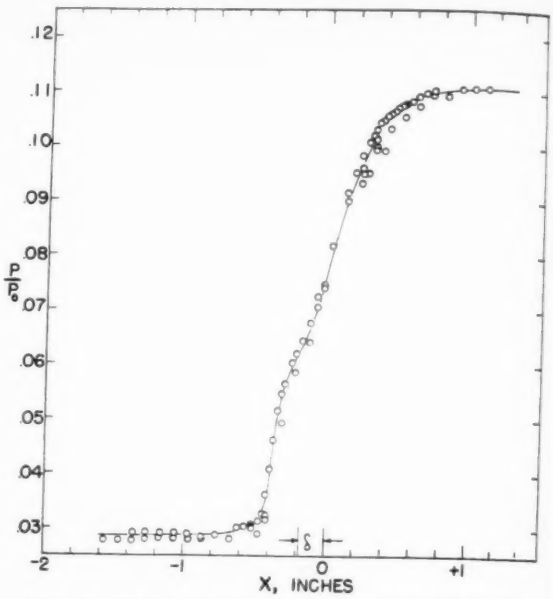


FIG. 13. Wall static pressure distribution for 10° shock.

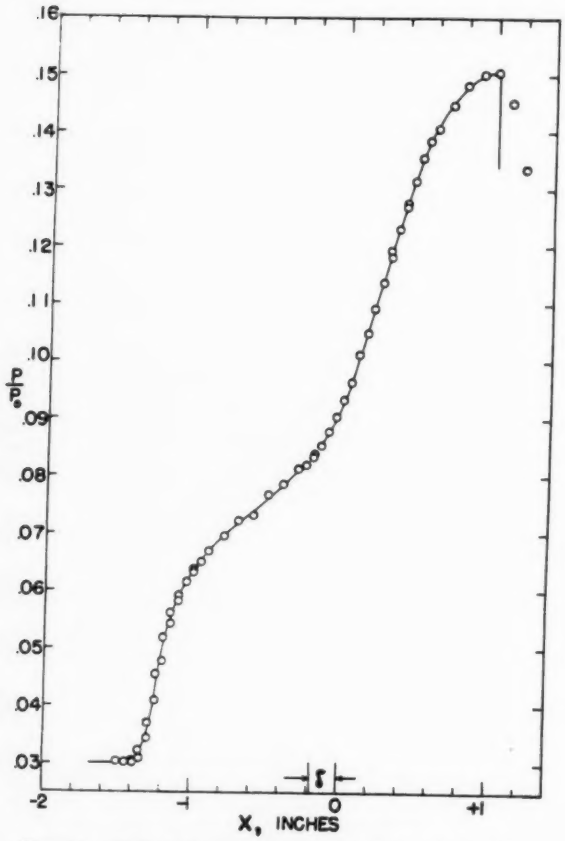


FIG. 14. Wall static pressure distribution for 14° shock.

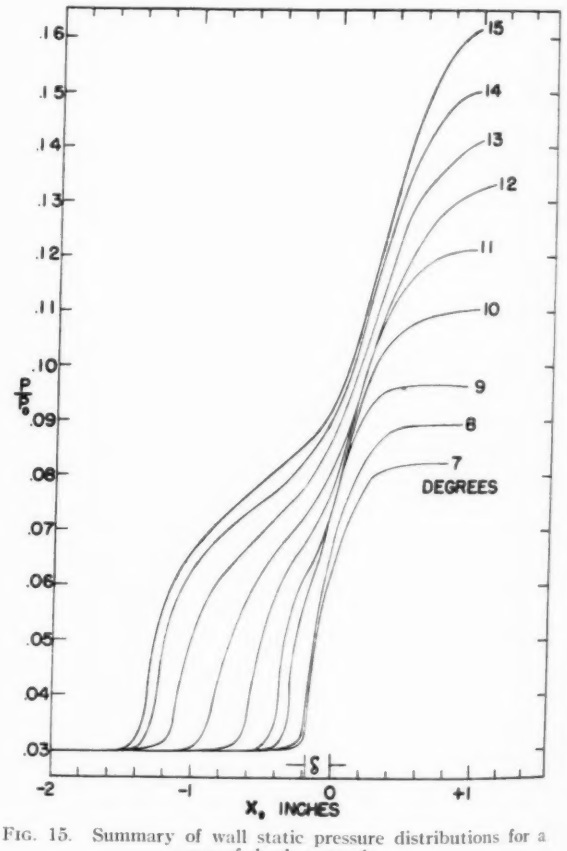


FIG. 15. Summary of wall static pressure distributions for a range of shock strengths.

into the stream, and more of the compression waves coalesce with it. The zero velocity line is inclined at an angle of approximately 8°. It is difficult from the total head measurements to differentiate between the edge of the mixing region and the strong compression waves between it and the shock.

(3) At six to eight boundary-layer thicknesses from the beginning of the interaction, the mixing region, the essentially uniform external flow, and the shock are well separated and defined. Using the pressure relationship across the shock in this region (eight boundary-layer thicknesses), the static pressure behind the shock was found to be about 5 per cent less than that measured on the wall.

(4) There is a continuous pressure rise along the wall up to about the eight boundary-layer thicknesses station, at which point the zero velocity line is approximately one boundary-layer thickness from the surface. Beyond this point there is a small pressure decrease, probably associated with the vortex in the corner. Because of this, no further studies were made beyond this point. In the reverse flow part of the separated region, flows as high as $M = 0.3$ were found. They were highest at the last station studied and decreased as the separation point was approached from the downstream direction.

Shock-Wave Boundary-Layer Studies

In an attempt to determine how the phenomenon is affected by the method of imposing the gradient that causes separation, the previous results have been compared to those obtained by studying the impingement of a shock wave on a turbulent boundary layer on a flat wall. These results, obtained at the Gas Dynamics Laboratory of the James Forrestal Research Center, Princeton University, have been previously reported[11] and so will be reviewed only briefly. Typical pressure distributions on the wall for several shock strengths are shown in Figs. 12–14. A summary of the pressure distributions obtained for a series of incident shock strengths ranging from a pressure ratio of about 1.6 to 3.1 is shown in Fig. 15. The changes in the phenomenon with shock strength are noticeable and are summarized in Fig. 16. There are strong changes in wall pressure gradients until shock pressure ratios of the order of 3 are approached. Total head tubes were again used to survey the separated region and to indicate the beginning of separation. Some results for an extremely strong shock ($P_2/P_1 = 2.9$) are shown in Fig. 17. For all shock strengths above 9°, separation was found at a wall pressure ratio of about 2, approximately two boundary-layer thicknesses downstream of the beginning of the interaction. On the basis of the optical studies and the inflection in the static pressure curve for 9°, separation must also be experienced for this case. The separated region was, however, too small to be detected by the techniques used.

The development of the phenomenon may be described as follows (Fig. 18):

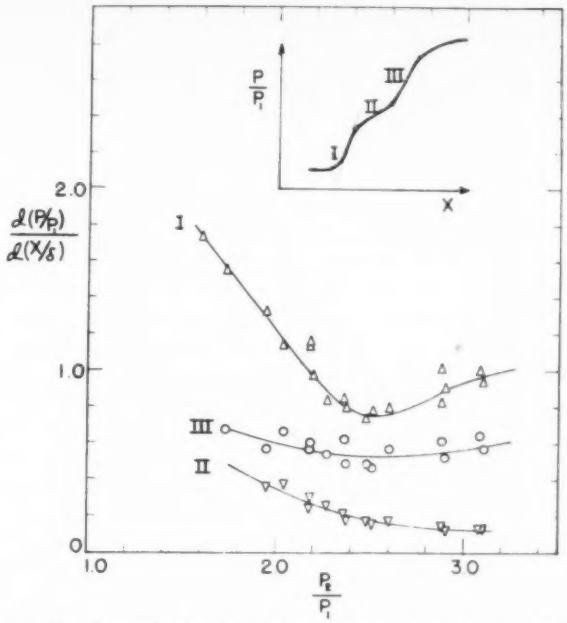


FIG. 16. Synopsis of wall static pressure gradients for varying incident shock pressure ratios.

(1) For weak shocks, there is no separation, the reflection is regular (single reflected compression wave), and the boundary layer is thinner after the interaction than before. This type of phenomenon is characteristic of the pressure distribution of Fig. 12. It should be pointed out that pressures of the order of 2.6 are reached without any indication of separation.

(2) For stronger shock waves, the beginning of separation is evidenced by a departure from the regular reflection pattern and a spreading out of the interaction region. The final boundary layer is still thinner than the original boundary layer, and the separated region is extremely small. This is characteristic of the pressure distribution of Fig. 13. The reflection has broken up into an initial compression, an expansion, and a final compression. For somewhat stronger shocks, the phenomena begin to spread out, and the final boundary layer is thicker than the initial one.

(3) For extremely strong shocks, the region of influence has become large, and the various com-

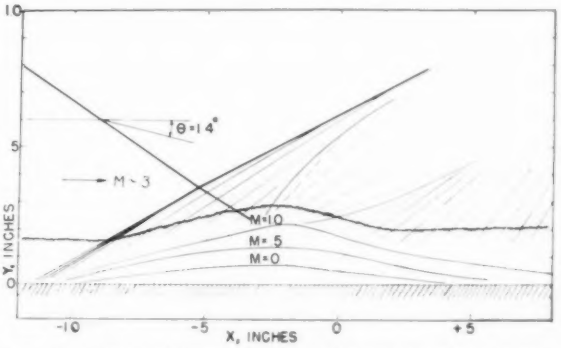


FIG. 17. Sketch of interaction for 14° shock showing the contours of constant Mach Number in the subsonic region.

ponents of the interaction phenomenon are well-defined and separated by relatively large distances. This is characteristic of the pressure distribution of Fig. 14. The separated region approaches a boundary-layer thickness in height, and the boundary layer after the interaction is thicker than the initial boundary layer.

To summarize these results: (1) Below an incident shock pressure ratio of about 1.7 (overall pressure ratio = 2.6) the interaction is similar to the theoretical inviscid case. The incident shock is reflected as a single compression, there is no separation, the pressure rise on the wall is smooth and complete in about two to three boundary-layer thicknesses. (2) Above shock strengths of 1.75 separation occurs about two boundary-layer thicknesses from the beginning of the interaction at a wall to initial static pressure ratio of about 2. The point of separation and the pressure ratio at separation are independent of shock strength over the range tested, although there are considerable changes in the phenomenon up to shock strengths of ~2.5. Above this point, the phenomenon appears to have become regular, growing only in size but not changing geometry with increased shock strengths.

Comparison of Separation Results

A comparison of the two detailed studies presented show several interesting phenomena are taking place.

(1) When the separated region is small, there seems to be a considerably different phenomenon than when the separation is large. Both the extremely small step and the weak shock show strong changes as the separation region grows in size until the height of the separated region approached a boundary-layer thickness.

(2) When the separated region becomes large, of the order of one boundary-layer thickness in height, both the step and incident shock results seem to approach relatively constant conditions where bigger steps and strong shocks are evidenced only by a change in the scale of the phenomenon but not in its geometry, pressure gradients, etc.

(3) It is interesting to compare the wall static pressure curves for one of the large steps with that obtained for a strong incident shock, and this is shown in Fig. 19. There is excellent agreement between the two sets of data up to seven or eight boundary-layer thicknesses from the beginning of the interaction. The separation point in both cases falls at the same pressure ratio and station. The detailed surveys through the separated region are also similar.

(4) The excellent agreement of (3) is obtained in spite of the considerable difference in the downstream flows. For the step, high normal gradients are experienced near the face of the step, and a vortex of some considerable strength is located in this region. For the shock-wave boundary-layer interaction, no such normal gradient has been found, but the reflection of the impinging shock from the separated layer results in a considerable change in flow direction and a subsequent large rise in pressure. It appears, therefore, that the separation and the associated phenomenon after the separation point are relatively independent of how the separation is caused. This is true for a region of the order of seven or eight boundary-layer thicknesses from the beginning of the phenomenon.

(5) It is of interest to examine the maximum pressure rise that can be obtained by this combination of separation plus the following mixing phenomenon. There seems to be a definite limit as evidenced by the maximum value reached for large step heights. This pressure ratio, of the order of 2.6 for the step, is about the same value that was reached for the extremely strong shock boundary-layer interaction just before the incident shock struck the separated boundary layer. It would be interesting to see further work at higher shock strengths. On the basis of this investigation, the pressure ratio should level off at approximately 2.6, and there should be no further increase until the shock strikes the layer.

(6) The results of (4) and (5) indicate that the pressure rise obtained after the flow separation depends, in some way, on the height of the separated region. Some preliminary results on another study indicate that lower peak pressures are obtained if the wall after the separation point curves away from the main flow. This moves upstream the point at which the separated region is about one boundary layer thick, cutting down the length available for mixing.

Reynolds Number Effects

Little data are directly available on the effect of Reynolds Number on separation phenomenon. However, reference 21 refers to results obtained by Lange on the effect of Reynolds Number on peak pressure ahead of a step. These results are shown in Fig. 20. In reference 16, the normal shock studies were made on two boundary layers that differed in Reynolds Number by a factor of 2. The results seem to indicate that over a fairly wide range of Reynolds Numbers there are little or no effects. This should be expected since the peak pressure is a combination of separation and mixing phenomena. Since the studies are concerned only with turbulent layers, it seems reasonable to expect no strong change in either of these phenomena with variable Reynolds Number.

Mach Number Effects

Fig. 10 shows the change of influence region of a disturbance with Mach Number. As the Mach Number decreases, there would be less spreading out of the phenomena due to the propagation of the disturbances perpendicular to that wall. However, with decreased Mach Number, the subsonic region increases in height. The propagation along the wall of a disturbance should be some function of this subsonic height. In addition, there are, of course, considerable changes in mixing rates with Mach Number. De-

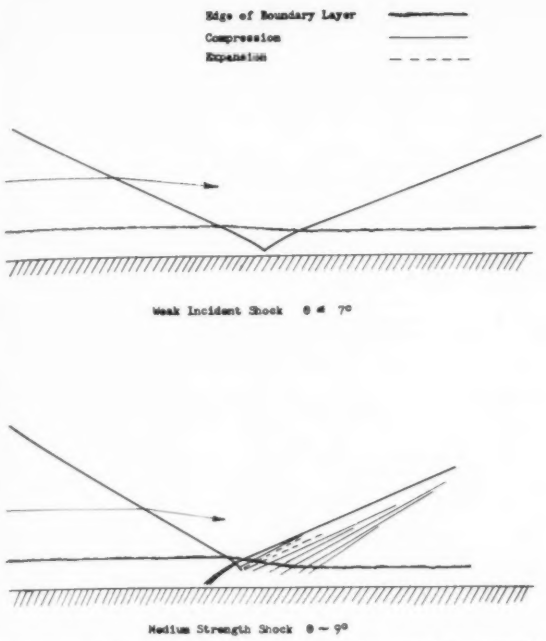


FIG. 18. Sketch of interaction region with varying strength shocks.

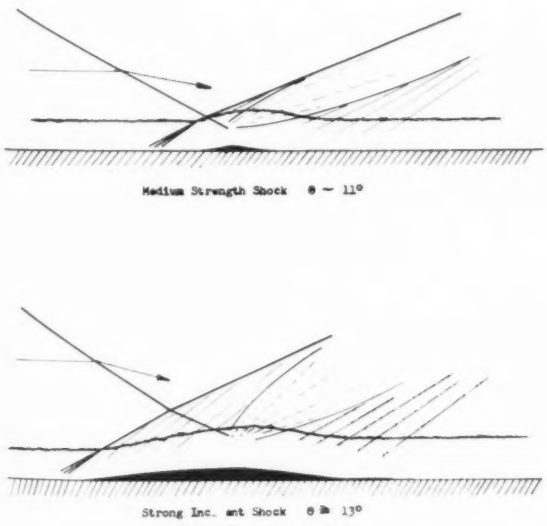


FIG. 18. Concluded.

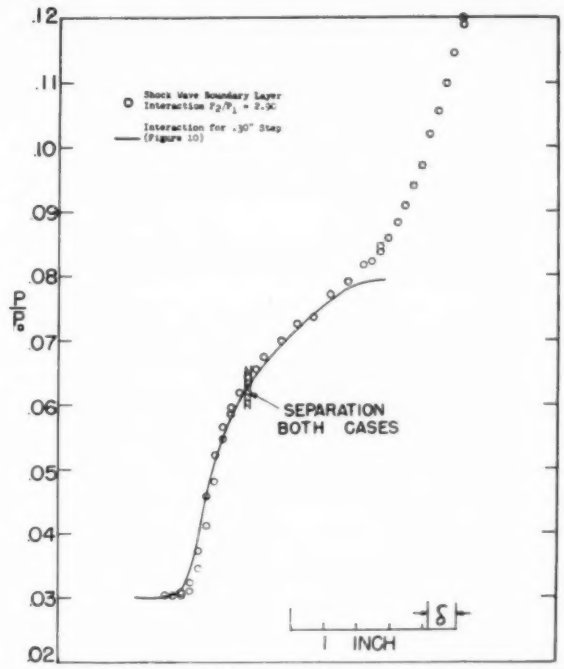


FIG. 19. Comparison of the wall static pressure curves for shock-wave boundary-layer interaction and step studies.

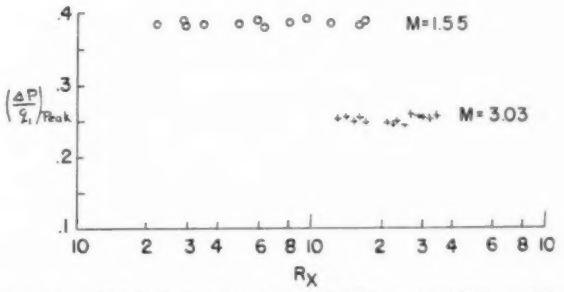


FIG. 20. NACA results of variation of $\Delta P/q$ peak with Reynolds Number for flow over a step.

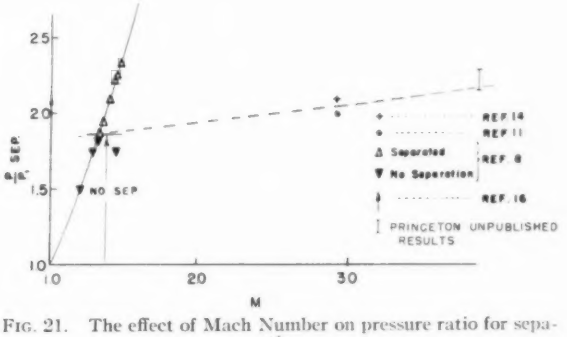


FIG. 21. The effect of Mach Number on pressure ratio for separation.

tailed data on the separation point are available only from references 8, 11, 14, and 16. The results of these four studies are summarized in Fig. 21. The work at low Mach Numbers by Fage and Sargent and by Ackeret, Feldmann, and Rott used normal shock waves at varying Mach Numbers to impose the pressure gradient. The work of the latter group did not extend far enough to obtain separation, but their results defined a definite region in which no separation occurs. The work of Fage and Sargent agrees well with these results and, in addition, has results in the separated region. These two studies combined with the work described herein show that the separation of a turbulent boundary layer is not an extremely strong function of Mach Number. At a Mach Number of 1.3 to 1.4, the boundary layer separates at a pressure ratio of 1.9 to 2.0, while at a Mach Number of 2.9, the separation pressure ratio is 2.0 to 2.1. These results indicate that the actual occurrence of separation is not primarily affected by the supersonic part of the boundary layer, since the energy in this region varies considerably and depends on the free-stream Mach Number. The separation phenomenon more likely depends on the subsonic-transonic lower region of the boundary layer which is always present and independent of free-stream conditions.

Conclusions

The results of the two investigations described and of other published results lead to the following conclusions.

(1) When the separated region is small, the phenomenon appears to be different from when the separated region is large. For small regions, stronger gradients may be experienced and large changes obtained with small changes in the disturbance. The phenomenon is associated with extremely strong gradients along and perpendicular to the wall. It seems likely that under such conditions the mixing phenomenon may differ widely from those usually expected under normal conditions. The basic parameters are as yet unknown, but obviously the gradients normal to the wall have a strong effect. This invalidates the use of the ordinary boundary-layer equations in this region.

(2) The occurrence of separation of a turbulent boundary layer from a flat wall, at least for the two cases tested, appears independent of how the strong gradient is imposed. At Mach Numbers near 3, deceleration to the separation point at a pressure ratio between 2 and 2.1 occurs in two to three boundary-layer thicknesses and is unaffected by the flow downstream.

(3) The occurrence of separation is not indicated by a cessation of pressure rise. After the separation point, a considerable pressure rise occurs for distances of the order of five to six boundary-layer thicknesses. The gradient decreases after the separation point, the pressure reaching a maximum approximately seven or eight boundary-layer thicknesses from the beginning of the interaction. This peak pressure may be associated with the fact that the zero velocity line in the separated region has reached a position approximately one boundary-layer thickness from the surface.

The maximum or peak pressure that can be obtained depends on a combination of separation plus a mixing phenomenon in the separation region. Of the order of 20 per cent of the total pressure rise occurs in the separated flow at $M = 3$. There is some indication that this part of the pressure rise decreases with Mach Number.

(4) The pressure ratio for separation has been found to be almost constant with Mach Number varying from 1.9 to 2 at $M \backsim 1.35$ and 2.0 to 2.1 at $M \backsim 2.9$. This corresponds to a $\Delta p/q$ of 0.7 for the low Mach Number case and 0.2 for the higher Mach Number case.

(5) The available results on Reynolds Number effects show extremely small or negligible effects on peak pressure ratios obtained ahead of steps. It seems likely that the separation pressure ratio will also be relatively unaffected by Reynolds Number variation.

(6) The detailed model shows that at the separation point strong normal gradients are experienced. With significant pressure rises occurring after separation, the only theoretical approach that seems applicable to the overall phenomena appears to be that of Crocco and Lees. Considerably more information on mixing rate and how it is influenced by the strong normal and longitudinal gradients must be obtained, however, before the theory can be applied to a wide variety of cases.

References

[1] Lees, L., *Interaction Between the Laminar Boundary Layer over a Plane Surface and an Incident Oblique Shock Wave*, Princeton University Aeronautical Engineering Dept. Rept. 143, January, 1949.

[2] Lighthill, M. J., *On Boundary Layers and Upstream Influence; Part I, A Comparison Between Subsonic and Supersonic Flows*, Proceedings of the Royal Society, Volume 217, 1953.

[3] Lighthill, M. J., *On Boundary Layers and Upstream Influence; Part II, Supersonic Flows with Separation*, Proceedings of the Royal Society, Volume 217, 1953.

[4] Crocco, L., and Lees, L., *A Mixing Theory for the Interaction Between Dissipative Flows and Nearly-Isentropic Streams*, Princeton University Aeronautical Engineering Dept. Rpt. 187, January, 1952.

[5] Leipmann, H. W., Roshko, A., and Dhawan, S., *On the Reflection of Shock Waves from Boundary Layers*, NACA TN 2334, April, 1951.

[6] Barry, F. W., Shapiro, S. H., and Neumann, E. P., *The Interaction of Shock Waves with Boundary Layers on a Flat Surface*, Meteor Report No. 52, MIT, March, 1950.

[7] Bardsley, O., and Mair, W. A., *The Interaction Between an Oblique Shock Wave and a Turbulent Boundary Layer*, Philosophical Magazine, Series 7, Vol. 42, January, 1951.

[8] Fage, A., and Sargent, R. E., *Shock Wave and Boundary Layer Phenomena Near a Flat Surface*, Proceedings of the Royal Society, Vol. 190, Series A, No. 1020, June, 1947.

[9] Bogdonoff, S. M., and Solarski, A. H., *A Preliminary Investigation of a Shock Wave-Turbulent Boundary Layer Inter-*

(Continued on page 430)

Laminar Heat Transfer in Two-Dimensional Subsonic Effusors*

PAUL A. LIBBY† AND MARIAN VISICH, Jr.‡

Polytechnic Institute of Brooklyn

SUMMARY

The laminar heat transfer in two-dimensional subsonic effusors such as are of interest in hypersonic wind tunnels is studied. The laminar boundary-layer equations for flows with axial pressure gradient and heat transfer are solved following the extension of the von Kármán method presented in reference 1. Fourth- and fifth-degree polynomials in the velocity and stagnation enthalpy profiles are used. A qualitative discussion of the influence of the axial pressure gradient on local heat transfer is followed by the calculation of the heat transfer for two contours, one more or less conventional and the other one which was expected to have reduced heat-transfer rates in the throat region. This expectation is borne out.

INTRODUCTION

THE EFFUSORS OF HYPERSONIC WIND TUNNELS, which involve operating conditions of high stagnation pressures and temperatures and structural requirements of small thermal and load deformations, are subjected to large rates of heat transfer. Since little is known about transition from laminar to turbulent boundary layers under the conditions of large axial accelerations and large rates of heat transfer existing in such effusors, it is not possible to predict whether laminar or turbulent heat-transfer rates should be applied. For conservative design purposes it is clear that turbulent rates should be used. However, little appears to be known about turbulent boundary layers with axial pressure gradient and large rates of heat transfer.

On the other hand recent interest in compressible laminar boundary layers with axial pressure gradient and heat transfer has been stimulated by the problem of aerodynamic heating in high-speed flight and has resulted in the development of a method of analysis[1] which leads to fairly accurate rates of laminar heat transfer without excessive computational difficulty. The application of this method to the subsonic effusor problem, despite its present restriction to laminar flow, is of interest in that contours leading to reduced rates of laminar heat transfer may be found and may infer reduced turbulent heat-transfer rates at least for the throat region where it is most desirable to reduce the thermal deformations and therefore the local heat transfer. Such modifications of the subsonic effusor contour are justified from a design point of view since their aerodynamic function—namely, to accelerate the flow from stagnation conditions to uniform sonic flow at the throat—can apparently be realized by a large number of greatly different contours. Therefore, it may be appropriate to design the effusor from an optimum heat-transfer point of view.

In this paper the method of reference 1 has been applied to the analysis of the laminar compressible boundary layer in a two-dimensional effusor. The method, which is based on an extension of the von Kármán integral analysis along the lines of Kalikhman,[2] is outlined here; certain modifications found to be desirable from a computational point of view as a result of the work presented here have been made, but all significant details may be found in reference 1.

BASIC EQUATIONS

Fig. 1 shows schematically the system under consideration. A two-dimensional effusor of typical contour is attached to a settling chamber in which exist specified stagnation pressure (p_s) and temperature (T_s). A s-n coordinate system with origin located at the attachment point of the effusor and the walls of the settling chamber is used to locate points within the boundary layer. The effusor accelerates the flow from stagnation conditions to a local Mach Number of unity at the throat while the supersonic nozzle downstream of the throat further accelerates the flow to the desired test section conditions. As in all boundary-layer calculations, the potential flow characteristics along s starting from the stagnation point ($s = 0$) are assumed known.

The basic partial differential equations describing the two-dimensional laminar compressible boundary layer of a perfect gas with a constant Prandtl Number of unity, a Chapman viscosity-temperature relation, and a constant coefficient of specific heat are:

$$\rho u u_s + \rho v u_n = -p_s + (\mu u_n)_n \tag{1}$$

$$p_n = 0; \quad -p_s = \rho_1 U_1 (dU_1/ds) \tag{2}$$

$$(\rho u)_s + (\rho v)_n = 0 \tag{3}$$

$$\rho u H_s + \rho v H_n = (\mu H_n)_n \tag{4}$$

$$p = \rho R T \tag{5}$$

$$\mu = C\mu_0 (T/T_0) \tag{6}$$

where

s = coordinate along surface
n = coordinate normal to surface

Received December 11, 1953. Revised and received January 17, 1955.

* This study was conducted under the sponsorship of the Air Research and Development Command on contract AF 18(600)-655. The authors are indebted to Professor Antonio Ferri for suggesting the study and for helpful discussions.

† Associate Professor of Aeronautical Engineering.

‡ Research Associate.

ρ = mass density of gas
u = velocity component in s direction
v = velocity component in n direction
p = static pressure
μ = coefficient of viscosity
$H = (u^2/2) + c_pT$, local stagnation enthalpy
R = gas constant
T = static temperature
$C = \sqrt{T_w/T_0}\,(T_0 + S)/(T_w + S)$, Chapman constant
S = Sutherland constant, 216°R. for air

and where the subscript zero denotes some arbitrary reference conditions and the subscript w denotes conditions at the wall. The assumptions of Prandtl Number unity and of linear viscosity temperature relation modified after Chapman have been used extensively in boundary-layer analyses and have been considered satisfactory for many engineering calculations.

Eqs. (1)–(6) can be formulated for approximate solution by an extension of the von Kármán integral technique; that is, Eqs. (1) and (4) are multiplied by dn and integrated from the wall to the edge of the boundary layer $\delta = \delta(s)$. By applying the continuity equation [Eq. (3)] and the Dorodnitzin transformation* and by rearrangement, the following two integral differential equations are obtained:

$$(F_1/2)\Lambda' + F_1\Lambda[\ln(F_1G)]' + F_2\Lambda[\ln(U_1/U_0)]' = [(u/U_1)_\tau]_w \quad (7)$$

$$(F_3/2)\Lambda' + F_3\Lambda[\ln(F_3G)]' = [(H/H_1)_\tau]_w \quad (8)$$

where

$$F_1 \equiv \int_0^1 (u/U_1)\,[1 - (u/U_1)]\,d\tau$$

$$F_2 \equiv [1 + (\gamma - 1)\,(M_1^2/2)] \int_0^1 [(H/H_1) - (u/U_1)^2]\,d\tau$$

$$F_3 = \int_0^1 (u/U_1)\,[1 - (H/H_1)]\,d\tau$$

$$\Lambda \equiv \rho_1 U_1 \rho_0 U_0 \delta_t^2/\mu_1\mu_0$$

$$G^2 \equiv \rho_1 U_1 \mu_1/\rho_0 U_0 \mu_0$$

$$\tau \equiv t/\delta_t$$

and where $\delta_t = \delta_t(s)$, the boundary-layer thickness in the Dorodnitzin s-t plane, γ = ratio of specific heats, M_1 = potential flow Mach Number, and $(\ )'$ denotes differential with respect to $\xi \equiv \rho_0 U_0 s/\mu_0$ and the subscript unity denotes quantities in the potential flow at any value of s.

Continuing the integral method, one now solves Eqs. (7) and (8) approximately by assuming profiles for u/U_1 and H/H_1 in terms of τ, satisfying boundary conditions at the surface ($\tau = 0$) and at the outer edge of the boundary layer ($\tau = 1$). With these profiles, the F integrals can be evaluated in terms of Λ and one other parameter in the assumed profiles to be determined by the differential equations.

In reference 1 sixth- and seventh-degree polynomials were assumed for the velocity and stagnation enthalpy profiles, since it was planned to use these for laminar boundary-layer stability calculations which require great accuracy in the profiles. For heat-transfer calculations fourth- and fifth-degree polynomials are usually considered satisfactory and are therefore used here.

Thus the velocity and stagnation enthalpy profiles are taken to be

$$u/U_1 = (2\tau - 2\tau^3 + \tau^4) + (z/3)\,(\tau - 3\tau^2 + 3\tau^3 - \tau^4) \quad (9)$$

and

$$H/H_1 = 1 - (1 - W)\,(1 - 10\tau^3 + 15\tau^4 - 6\tau^5) + b_1(\tau - 6\tau^3 + 8\tau^4 - 3\tau^5) \quad (10)$$

where

$$z \equiv (W\Lambda/2M_1)\,(dM_1/d\xi)$$

$$W \equiv H_w/H_1$$

and where b_1 is taken as the additional parameter to be determined along with Λ from the solution of Eqs. (7) and (8). Examination of Eq. (10) indicates that physically b_1 is the slope of the stagnation enthalpy profile at the wall in the s-t plane. The parameter W by definition gives the wall temperature in per cent of stagnation temperature and, since $N_{pr} = 1$, as a per cent of recovery temperature. It is pointed out that with the restriction to fourth- and fifth-degree polynomials the wall temperature may be considered variable without great difficulty. This was not the case in reference 1, where the higher degree polynomials and their associated additional boundary conditions were significantly simplified by the assumption of constant wall temperatures.

In the effusor with water or air cooling, the real heat-transfer problem is most closely approximated by relating the wall temperature to the heat transfer-rate—that is, by relating b_1 to W. Although in principle this involves no computational difficulty, in the interest of simplicity the wall temperature has been assumed constant here. This assumption implies a variation of coolant temperature along s which may not be in accordance with the cooling passage design. All that can be said of this assumption in general is that in regions where the real coolant is at a lower temperature than that indicated by the assumption of constant wall temperature, the wall will be cooler and heat transfer higher. This higher rate will lead to actual rates of

* The Dorodnitzin transformation involves the definition of a new normal coordinate t such that for a given s, $dt = (\rho/\rho_1)\,dn$ and thus the introduction of a boundary-layer thickness in the s-t plane given by

$$\delta = \int_0^{\delta_t} (\rho_1/\rho)\,dt = \int_0^{\delta_t} (T/T_1)\,dt.$$

Since skin friction and heat transfer can be calculated from the boundary-layer characteristics in the s-t plane, the inverse transformation must only be performed if the physical boundary-layer thickness or profiles are required.

heat transfer downstream less than predicted on the assumption of constant wall temperature even if corrected by making the heat-transfer rate proportional to $(1 - W)$—i.e., by applying the constant wall temperature results to local conditions. Similar considerations can be made when the coolant temperature is higher than indicated by this constant wall temperature analysis.

With the profiles of Eqs. (9) and (10), the F integrals become

$$F_1 = (37/315) - (2z/945) - (z^2/2268) \tag{11}$$

$$F_2 = [1 + (\gamma - 1)(M_1^2/2)][(263/630) - (1/2)(1 - W) - (71z/3780) - (z^2/2268) + (b_1/10)] \tag{12}$$

$$F_3 = (1 - W)[(17/70) + (31z/2520)] - (79b_1/1260) - (19b_1z/7560) \tag{13}$$

In principle Eqs. (7) and (8) can now be integrated from initial conditions at $\xi = 0$ with the prescribed variation of potential flow properties with ξ and thus give $\Lambda = \Lambda(s)$ and $b_1 = b_1(s)$ even for variable W. From this solution the heat transfer can be determined; thus

$$q \equiv (kT_n)_w$$

becomes in terms of the variables pertinent here

$$q = \sqrt{C}\rho_0 U_0 H_1 (b_1/\sqrt{\Lambda})\sqrt{M_1/M_0}\{[1 + (\gamma - 1)(M_0^2/2)] \div [1 + (\gamma - 1)(M_1^2/2)]\}^{(3\gamma - 1)/[4(\gamma - 1)]} \tag{14}$$

If the boundary-layer thickness if of interest, then from definitions of the Dorodnitzin transformation and of δ_0 it may be found

$$\rho_0 U_0 \delta/\mu_0 = (C\sqrt{\Lambda}/G)\{[1 + (\gamma - 1)(M_0^2/2)]/[1 + (\gamma - 1)(M_1^2/2)]\}[F_2 - F_1 + (7/10) + (z/60)] \tag{15}$$

However for W = constant it is found computationally convenient to modify this procedure and to treat M_1 as the independent variable and z and b_1 as the dependent variables. These changes lead to the following form of the equations:

$$M_1[(37/315) - (6/945)z - (5/2268)z^2](dz/dM_1) = 2W + z\{[(1/21) - (2W/3) + (5z/126) + (z^2/756) - (b_1/5)] - [(37/315) - (2z/945) - (z^2/2268)]K\} \tag{16}$$

$$2M_1(dF_3/dM_1) = (b_1 W/z) - F_3[K + 1 + (M_1/z)(dz/dM_1)] \tag{17}$$

where

$$K \equiv M_1\kappa + \{(1 - \gamma M_1^2)/[1 + (\gamma - 1)(M_1^2/2)]\}$$
$$\kappa \equiv (d^2\xi/dM_1^2)/(d\xi/dM_1)$$

and where Eq. (13) relating b_1 and F_3 is required to complete the system of equations.

It is of interest to note that in Eqs. (16) and (17), which follow from Eqs. (7) and (8) with only the assumption of the velocity and stagnation enthalpy profiles and of constant W, the potential flow characteristics are entirely contained in the quantity κ which must be specified as a function of M_1.

Eqs. (16) and (17) may be integrated from the initial conditions on M_1 and used to determine z and b_1 as functions of M_1; since $M_1 = M_1(\xi)$ is known, z and b_1 can be found as a function of ξ. For the effusor problem considered here, the initial conditions correspond to that of a stagnation point where $M_1 = 0$. The requirement that the solution be well-behaved to the limit as $M_1 \to 0$ requires that b_1 and z take on special values obtained from consideration of Eqs. (16) and (17). It can be seen from these equations that if dz/dM_1 and dF_3/dM_1 are to remain finite as $M_1 \to 0$,

$$2W + z_s\{[(1/21) - (2W/3) + (5z_s/126) + (z_s^2/756) - (b_{1s}/5)] - [(37/315) - (2z_s/945) - (z_s^2/2268)]K_s\} = 0 \tag{18}$$

$$(b_{1s} W/z_s) - F_3(1 + K) = 0 \tag{19}$$

where the subscript s is used to denote the special stagnation point values of z, b_1, and K. Eqs. (18) and (19) are two equations in two unknowns, z_s and b_{1s}.

Eqs. (16), (17), and (13) can be handled in a straight forward manner from these initial conditions; only the parameters W, κ, and γ must be specified. The numerical integration may be performed by any standard method for first-order simultaneous ordinary differential equations. It may, however, be necessary for general κ to resort to a power series expansion of z and b_1 to get away from the origin. The necessity for this is eliminated if the potential flow in the neighborhood of the stagnation point can be represented by

$$s = kM_1^n \tag{20}$$

where k and n constants. Then for $0 \leq M_1 \ll 1$

$$K = K_s = n \tag{21}$$

and $z = z_s$ and $b_1 = b_{1s}$; that is, the stagnation values of z and b_1 satisfy the differential equations [Eqs. (16) and (17)] for values of M_1 in the neighborhood of $M_1 = 0$ and the conditions of finiteness of dz/dM_1 and $dF_3 \div dM_1$ at $M_1 = 0$ for potential flow distributions given by Eq. (20). This permits the numerical solution to be started away from the stagnation point. It is pointed out that potential flow distributions given by Eq. (20) correspond to the wedge type flows treated extensively in laminar boundary-layer literature.

It is pointed out that in handling Eqs. (16) and (17) the customary restrictions on z—namely, that $-6 < z < 6$— dictated by the requirements respectively of positive shear stress at the wall and monotonically increasing velocity profiles must be acknowledged. Any $\kappa = \kappa(M_1)$ function leading to values of z outside this range are usually not considered admissable.

General Considerations

On the basis of past experience with the integral equation approach and of an examination of Eq. (14), several points of interest with respect to the nozzle heat transfer may be established. It has been found that with a pressure gradient the moderate acceleration usually encountered in effusors the quantities b_1 and z do not vary greatly. Indeed, the slow variation of these variables has been used as a basis for the approximate solution of the equations corresponding to Eqs. (16) and (17) by Kalikman[2] and Morduchow.[3] Therefore, for M_1 on the order of unity, it is easily seen from Eq. (14) that for a given W and for given stagnation conditions

$$q\alpha(dM_1/ds)^{1/2} \tag{22}$$

This statement is clear physically; the local heat transfer depends primarily on the thickness of the boundary layer. Moreover, in the presence of negative pressure gradients, the boundary-layer thickness increases, remains constant, or decreases, depending upon the magnitude of the gradient. Eq. (22) indicates that to keep q small in a given region the acceleration of the flow in that region must be kept as small as possible. If one-dimensional flow considerations are applied to the contraction section of the nozzle, one concludes that for a given length in the axial direction the contour should be as flat as possible in the neighborhood of the throat if the throat heat transfer is to be reduced. Thus, for example, the contour denoted by b in Fig. 2 should be expected to yield lower local heat-transfer rates than contour a. It is pointed out that the contours are assumed to have the same throat location; it is clear that for reduced maximum local rates of heat transfer the nozzle should be as long as possible.

In the following section this hypothesis is examined by applying Eqs. (16) and (17) to the determination of the heat transfer in the two contours shown in Fig. 2.

Numerical Examples

The potential flow Mach Number distribution in the two effusors denoted by a and b was determined by one-dimensional flow considerations for those portions wherein the slope of the contour was small compared to unity. This corresponds to a local Mach Number of about 0.1 for contour a and 0.2 for contour b. A potential flow distribution corresponding to Eq. (20) was assumed from the stagnation point to these local Mach Numbers with k and n chosen so that the $s = s(M)$ function and its first derivative were continuous. This results in a discontinuous κ distribution which leads to no computational difficulties since the numerical solution is started at the discontinuity and which appears physically acceptable since it corresponds to a discontinuity in the contour curvature. The values of $k = 9.28$ and $n = 0.122$ were obtained for contour a and $k = 7.07$ and $n = 0.0784$ for contour b.

In Fig. 3 are shown the potential flow Mach Number gradients for the contours presented in terms of dM/ds with units of 1/inch. This dimensional representation was used since, in the final determination of q from Eq. (14) with Λ expressed in terms of z and dM/ds, the units of the reciprocal of the square root of a length appear on both sides of the equation. While the reference quantities denoted by the subscript zero could have been left in the equation, it was considered desirable to indicate the independence of q from those quantities. The discontinuity in the κ for contours a and b at local Mach Numbers of 0.1 and 0.2, respectively, can be seen from Fig. 3. It may be remarked that this discontinuity is small for contour a and that the results for contour b were found to remain essentially unchanged if a dM/ds distribution making κ continuous was assumed for $0 < M_1 < 0.2$. For this second calculation of contour b, the polynomial expansion in the neighborhood of the stagnation point mentioned previously was used to initiate the numerical integration.

In Fig. 4 the results of the numerical integration of Eqs. (16) and (17) for the determination of z and b_1 for the two contours are given. It will be noted that the integration for contour a has been continued into the supersonic portion of the effusor while that for contour b has been terminated at the throat. This latter restriction was necessitated by the discontinuity in the contour curvature which exists at the throat of contour b and which, therefore, yields a discontinuity in dM/ds. Since a discontinuity in z is not physically reasonable, the continuation of the solution beyond the throat would require additional parameters in the velocity and stagnation enthalpy profiles to make the discontinuity tractable, as in reference 4.

In Fig. 5 the heat-transfer distribution presented in terms of stagnation conditions but dimensionally consistent with dM/ds is shown for both contours. It will be noted that the peak heat transfer in contour b is only several per cent less than for contour a, but at the throat it is 36 per cent less and the overall heat transfer from the stagnation point to the throat is reduced by approximately 7 per cent. If the solution for contour b could have been continued beyond the throat, the overall heat transfer would have undoubtedly been reduced further percentagewise. It is pointed out that the boundary-layer thickness at the throat is calculated to be 60 per cent greater with contour b. Thus the expected advantage of contour b as indicated by the importance of dM/ds in determining the heat transfer is borne out by the numerical integration.

It can be expected that similar reduced rates for turbulent heat transfer would be realized with contour b since the turbulent layer would also be thicker at the throat, being more like a flat plate boundary layer than that which would develop on contour a. This expectation was borne out by calculations of the incompressible turbulent boundary layer with heat transfer and axial pressure gradient for both contours a and b. A reasonable transition Reynolds Number and typical hypersonic wind-tunnel stagnation conditions were assumed. The calculations indicated that

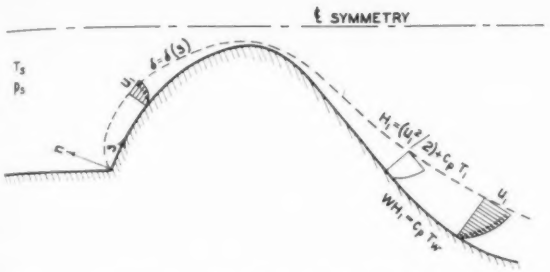


FIG. 1. Schematic drawing of two-dimensional effusor.

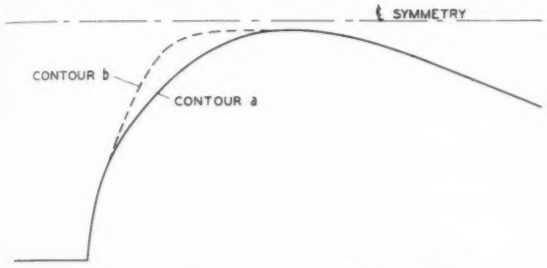


FIG. 2. Contours of two possible effusors.

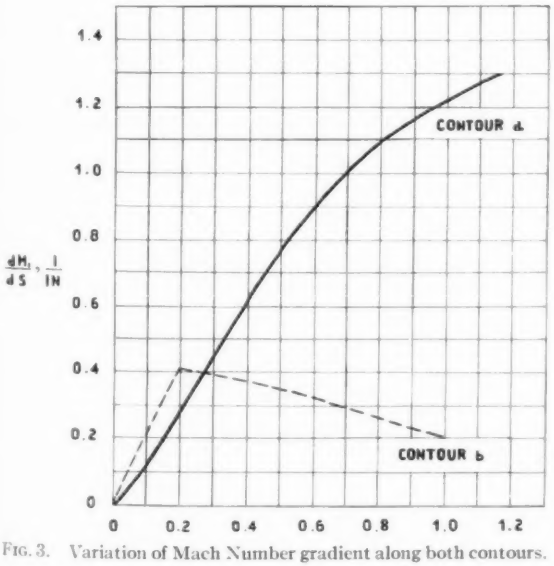


FIG. 3. Variation of Mach Number gradient along both contours.

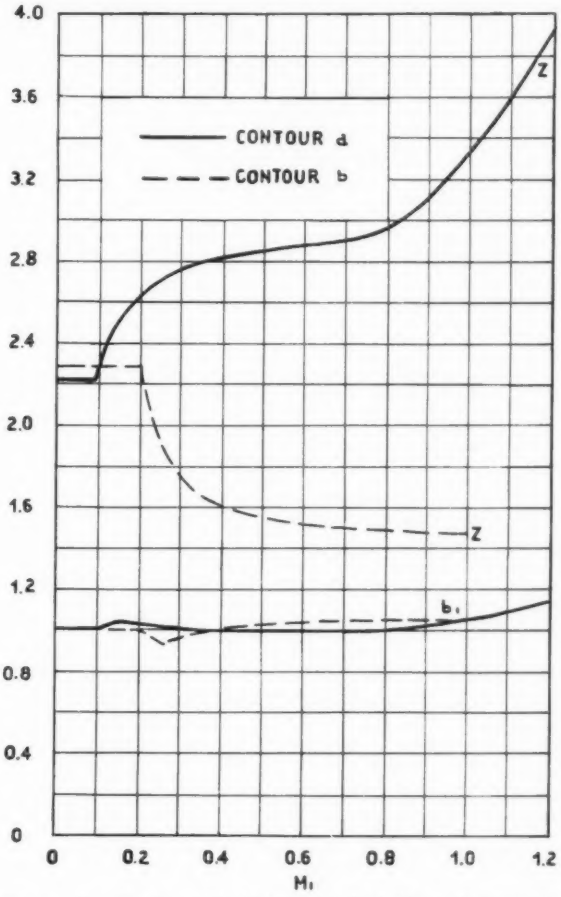


FIG. 4. Results of numerical integration for both nozzle contours.

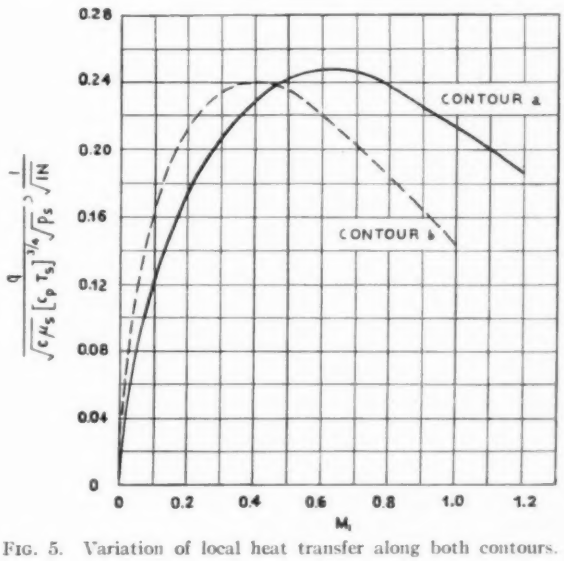


FIG. 5. Variation of local heat transfer along both contours.

the peak and throat local heat-transfer rates for contour *b* were less than those for contour *a* by 14 per cent and 26 per cent, respectively.*

Conclusions

By the use of the integral method with fourth- and fifth-degree polynomials for the velocity and stagnation enthalpy profiles, respectively, the two-dimensional, steady laminar boundary-layer equations for axial pressure gradient and heat transfer have been put in a form especially convenient for numerical integration. On the basis of some qualitative considerations, the heat transfer in two contours suitable for subsonic effusors have been compared, and a contour for reduced heat transfer at the throat has been suggested. Exact numerical integration of the differential equations giving the heat transfer for the two contours has been carried out and have confirmed the qualitative considerations.

* After submittal of this paper, the senior author learned that M. Sibulkin of the Jet Propulsion Laboratory of the California Institute of Technology has recently completed a study of the influence of the curvature of the effusor in the throat region on turbulent heat transfer. This study was recently published under the title *Heat Transfer to an Incompressible Turbulent Boundary Layer and Estimation of Heat Transfer Coefficients at Supersonic Nozzle Thorats*, Jet Propulsion Laboratory Report No. 20-78, 1954.

References

[1] Libby, Paul A., and Morduchow, Morris, *A Method for the Calculation of the Compressible Laminar Boundary Layer with Axial Pressure Gradient and Heat Transfer*, NACA TN 3157, January, 1954.

[2] Kalikhman, L. E., *Heat Transmission in the Boundary Layer*, NACA TN 1229, April, 1949.

[3] Morduchow, Morris, and Clarke, Joseph H., *A Method for the Calculation of Compressible Laminar Boundary Layer Characteristics in an Axial Pressure Gradient with Zero Heat Transfer*, NACA TN 2784, September, 1952.

[4] Libby, Paul A., and Pallone, Adrian, *A Method for Analyzing the Heat Insulating Properties of the Laminar Compressible Boundary Layer*, Journal of the Aeronautical Sciences; Vol. 21, No. 12, pp. 825–834, December, 1954.

Separation of a Supersonic Turbulent Boundary Layer

(*Continued from page 424*)

action, Princeton University Aeronautical Engineering Dept. Rpt. 184, 1951.

[10] Gadd, G. E., *Interactions Between Wholly Laminar or Wholly Turbulent Boundary Layers and Shock Waves Strong Enough to Cause Separation*, Journal of the Aeronautical Sciences, Vol. 20, No. 11, pp. 729–739, November, 1953.

[11] Bogdonoff, S. M., Kepler, C. E., and Sanlorenzo, E., *A Study of Shock Wave Turbulent Boundary Layer Interaction at M = 3*, Princeton University Aeronautical Engineering Dept. Rpt. 222, July, 1953.

[12] Donaldson, C. duP., and Lange, R. H., *Study of the Pressure Rise Across Shock Waves Required To Separate Laminar and Turbulent Boundary Layers*, NACA TN 2770 September, 1952.

[13] Beastall, D., and Eggink, E., Royal Aircraft Establishment, Technical Note 2041 (Restricted).

[14] Kepler, C. E., and Bogdonoff, S. M., *Interaction of a Turbulent Boundary Layer with a Step at M = 3*, Princeton University Aeronautical Engineering Dept. Rpt. 238, September 1, 1953.

[15] Moeckel, W. E., *Flow Separation Ahead of Blunt Bodies at Supersonic Speeds*, NACA TN 2418, July, 1951.

[16] Ackeret, J., Feldmann, F., and Rott, N., *Investigation of Compression Shocks and Boundary Layers in Gases Moving at High Speed*, NACA TM 1113, 1947.

[17] Barry, F. W., *A Review of Experimental Results on Boundary Layer-Shock Wave Interaction*, North American Aviation Rpt. No. AL-1599, 1952.

[18] Harrin, E. N., *A Flight Investigation of Laminar and Turbulent Boundary Layers Passing Through Shock Waves at Full-Scale Reynolds Numbers*, NACA TN 3056, December, 1953.

[19] Drougge, G., *An Experimental Investigation of the Influence of Strong Adverse Pressure Gradients on Turbulent Boundary Layers at Supersonic Speeds*, Aeronautical Research Institute of Sweden, Report No. 46, 1953.

[20] Lee, J. D., *The Influence of High Adverse Pressure Gradients on Boundary Layers in Supersonic Flow*, Institute of Aerophysics, University of Toronto, Report No. 21, October, 1952.

[21] Abbott, I. H., *Some Factors Contributing to Scale Effect at Supersonic Speeds*, NACA Contribution to the Fourth Meeting of AGARD Wind Tunnel and Model Testing Panel, September 5–11, 1953, London.

Readers' Forum

BRIEF REPORTS of investigations in the aeronautical sciences and discussions of papers published in the JOURNAL are presented in this special department. Publication is completed as soon as possible after receipt of the material. The Editorial Committee does not hold itself responsible for the opinions expressed by the correspondents.

A Note on Simplified Single Spool Turbojet Equilibrium Operation

Charles A. MacGregor
Aeodynamicist, Propulsion Research Corporation, Santa Monica, Calif.
January 6, 1955

Symbols

C_1, C_2, and C_3	= constants	***Subscripts***	
C_p	= specific heat at constant pressure	c	= compressor
N	= rotational speed	Des.	= design
P	= absolute total pressure	t	= turbine
T	= absolute total temperature	2, 4, 5, and 7	= see Fig. 1
W	= weight flow of gas		
γ	= ratio of specific heats		
η	= isentropic efficiency		

Introduction

THE ANALYSIS OF THE EQUILIBRIUM operation of a single spool turbojet engine having fixed geometry (see Fig. 1) may be simplified by considering that the turbine nozzle and some fixed area at the turbine exit are both choked. For most conditions, this assumption is valid because of the large pressure ratio across the jet nozzle. This assumption reduces the turbine operation to a point on its performance map.[1]

For a given compressor inlet temperature and pressure (flight Mach Number and altitude) and a constant jet nozzle area, the thrust will be a maximum for constant mechanical r.p.m. and combustion temperature operation. In general, this type of operation is not possible for this case, because there is only one independent variable. However, there is one compressor corrected airflow-r.p.m. characteristic with which this type of operation is possible. This note determines what the compressor corrected airflow-r.p.m. characteristics must be to permit this constant r.p.m., constant combustion temperature, and constant jet nozzle area operation.

Discussion

Consider a single spool turbojet engine consisting of an inlet, compressor, burner, turbine, and exhaust system (tailpipe) aerodynamically in series. Assume that the turbine nozzle is choked and that some area at the turbine exit is also choked and that this area remains constant; in addition, the pressure ratio across the burner is constant, and leakage air is equal to the amount of fuel added. Then

$$P_4/P_5 = \text{constant} = C_1 \tag{1}$$

$$W\sqrt{T_5}/P_5 = \text{constant} = C_2 \tag{2}$$

$$(T_5 - T_7)/T_5 = \text{constant} = C_3 \tag{3}$$

The equilibrium operation of the engine will be determined by satisfying two requirements—continuity of flow and equal compressor and turbine power.

From the continuity of flow between the compressor inlet and the turbine inlet and in terms of the values as compared to the design values, then

$$\frac{(P_4/P_2)}{(P_4/P_2)_{\text{Des.}}} = \frac{(W\sqrt{T_2}/P_2)}{(W\sqrt{T_2}/P_2)_{\text{Des.}}}\sqrt{\frac{(T_5/T_2)}{(T_5/T_2)_{\text{Des.}}}} \tag{4}$$

From the condition of equal turbine and compressor power and the definition of isentropic compressor efficiency, then

$$\frac{(P_4/P_2)}{(P_4/P_2)_{\text{Des.}}} = \left\{\frac{\left[\frac{\eta_c}{(\eta_c)_{\text{Des.}}}\right]\left[\frac{(T_5/T_2)}{(T_5/T_2)_{\text{Des.}}}\right] + \frac{C_{P_c}}{(C_{P_t})(C_3)[(\eta_c)_{\text{Des.}}][(T_5/T_2)_{\text{Des.}}]}}{1 + \frac{C_{P_c}}{(C_{P_t})(C_3)[(\eta_c)_{\text{Des.}}][(T_5/T_2)_{\text{Des.}}]}}\right\}^{\gamma/(\gamma-1)} \tag{5}$$

Combining Eqs. (4) and (5), then

$$\frac{(W\sqrt{T_2}/P_2)}{(W\sqrt{T_2}/P_2)_{\text{Des.}}} = \frac{1}{\sqrt{\frac{(T_5/T_2)}{(T_5/T_2)_{\text{Des.}}}}}\left\{\frac{\left[\frac{(\eta_c)}{(\eta_c)_{\text{Des.}}}\right]\left[\frac{(T_5/T_2)}{(T_5/T_2)_{\text{Des.}}}\right] + \frac{C_{P_c}}{(C_{P_t})(C_3)[(\eta_c)_{\text{Des.}}][(T_5/T_2)_{\text{Des.}}]}}{1 + \frac{C_{P_c}}{(C_{P_t})(C_3)[(\eta_c)_{\text{Des.}}][(T_5/T_2)_{\text{Des.}}]}}\right\}^{\gamma/(\gamma-1)} \tag{6}$$

Eq. (6) relates the corrected compressor inlet flow to the combustion temperature ratio. However, the corrected airflow characteristic of a compressor is determined by the corrected r.p.m. $(N/\sqrt{T_2})$ of the compressor. Since

$$\frac{(T_5/T_2)}{(T_5/T_2)_{\text{Des.}}} = \frac{\left[\dfrac{(N/\sqrt{T_2})}{(N/\sqrt{T_2})_{\text{Des.}}}\right]^2 \dfrac{(T_5)}{(T_{5\ \text{Des.}})}}{[(N)/(N_{\ \text{Des.}})]^2} \tag{7}$$

Then Eq. (6) becomes in terms of corrected r.p.m.,

$$\frac{\left(\dfrac{W\sqrt{T_2}}{P_2}\right)}{\left(\dfrac{W\sqrt{T_2}}{P_2}\right)_{\text{Des.}}} = \frac{\left[\dfrac{(N)}{(N)_{\text{Des.}}}\right]}{\left[\dfrac{(N/\sqrt{T_2})}{(N/\sqrt{T_2})_{\text{Des.}}}\right]\left[\dfrac{(T_5)}{(T_5)_{\text{Des.}}}\right]^{1/2}} \times \left\{\frac{\left[\dfrac{(\eta_c)}{(\eta_c)_{\text{Des.}}}\right]\dfrac{\left[\dfrac{(N/\sqrt{T_2})}{(N/\sqrt{T_2})_{\text{Des.}}}\right]^2\left[\dfrac{(T_5)}{(T_{5\ \text{Des.}})}\right]}{[(N)/(N_{\ \text{Des.}})]^2} + \dfrac{C_{P_c}}{(C_{P_T})(C_3)\ [(\eta_{c\ \text{Des.}})]\ [(T_5/T_2)/(T_5/T_{2\ \text{Des.}})]}}{1 + \dfrac{C_{P_c}}{(C_{P_T})(C_3)\ [(\eta_c)_{\text{Des.}}]\ [(T_5/T_2)/(T_5/T_{2\ \text{Des.}})]}}\right\}^{\gamma/(\gamma-1)} \tag{8}$$

This equation consists of two variable $(N)/(N)_{\text{Des.}}$ and $(T_5)/(T_{5\ \text{Des.}})$ for given boundary conditions and constants (assuming compressor airflow and efficiency are known functions of N). One of these variables will be independent and one variable will be dependent.

There is one particular corrected airflow-r.p.m. variation that will permit both constant $(N)/(N)_{\text{Des.}}$ and $(T_5)/(T_{5\ \text{Des.}}$ operation, neglecting the effect of compressor efficiency, which is small. This airflow variation is

$$\frac{\left(\dfrac{W\sqrt{T_2}}{P_2}\right)}{\left(\dfrac{W\sqrt{T_2}}{P_2}\right)_{\text{Des.}}} = \frac{1}{(N/\sqrt{T_2})/(N/\sqrt{T_2})_{\text{Des.}}} \left\{\frac{\left[\dfrac{(\eta_c)}{(\eta_c)_{\text{Des.}}}\right]\left[\dfrac{(N/\sqrt{T_2})}{(N/\sqrt{T_2})_{\text{Des.}}}\right]^2 + \dfrac{C_{P_c}}{(C_{P_T})(C_3)\ [(\eta_{c\ \text{Des.}})]\ [(T_5/T_{2\ \text{Des.}})]}}{1 + \dfrac{C_{P_c}}{(C_{P_T})(C_3)\ [(\eta_{c\ \text{Des.}})]\ [(T_5/T_{2\ \text{Des.}})]}}\right\}^{\gamma/(\gamma-1)} \tag{9}$$

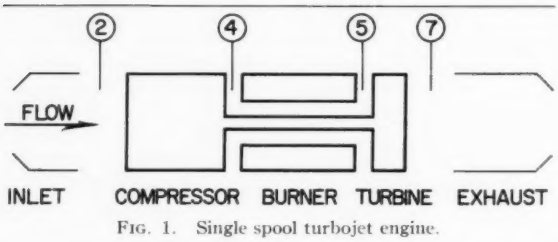


FIG. 1. Single spool turbojet engine.

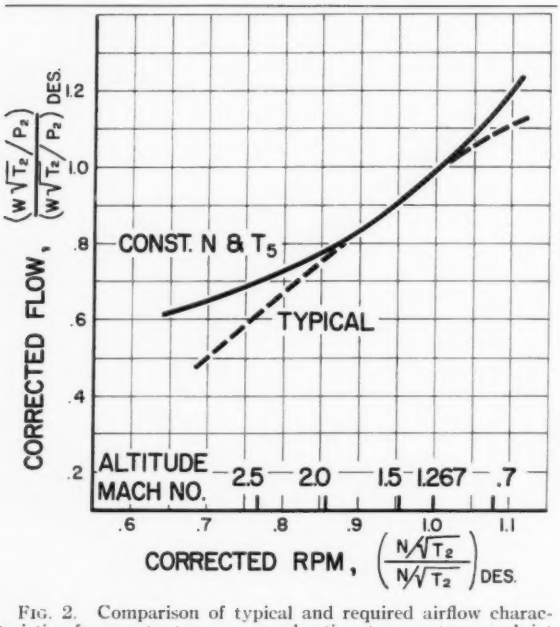


FIG. 2. Comparison of typical and required airflow characteristics for constant r.p.m., combustion temperature, and jet nozzle area operation.

Fig. 2 shows a comparison between this airflow characteristic and a typical compressor characteristic. It is seen that over the most interesting range this condition is reasonably well approximated and this type of operation is possible.

REFERENCE

[1] MacGregor, Charles A., *Flow Between Nozzles in Series with an Isentropic Work Change*, Journal of the Aeronautical Sciences, Vol. 21, No. 6, pp. 423-424, June, 1954.

Plasticity Coefficients for the Plastic Buckling of Plates and Shells

Alexander Krivetsky
Head, Analytical Methods Group, Structures Section, Bell Aircraft Corporation

November 15, 1954

ABSTRACT

The effect of plasticity on the buckling of plates, plate columns, or shells is considered by introducing a factor, called a plasticity coefficient, η, to allow for the departure of the stress-strain curve from the original straight elastic portion. Carpet plots are presented for the evaluation of the plasticity coefficients for five cases based on the original work of Bijlaard.* [1-5]

INTRODUCTION

THE DETERMINATION of the inelastic buckling stresses of plates and shells is a necessary requirement for the structural design of air frames. Formulas for computing the plasticity or reduction factors for the plastic buckling of plates and shells for various cases of loading and boundary conditions have been computed by Bijlaard.[1-5] The actual applications of these formulas for the evaluation of the plasticity coefficient and the

* The author wishes to acknowledge the helpful discussions with Paul P. Bijlaard, Professor, Cornell University, and Structures Consultant, Bell Aircraft Corporation, during the course of the work.

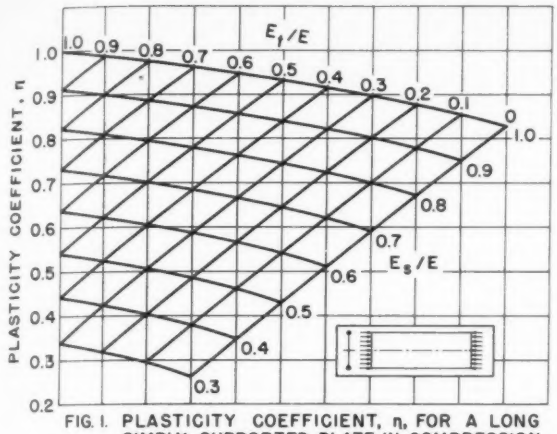


FIG. 1. PLASTICITY COEFFICIENT, η, FOR A LONG SIMPLY-SUPPORTED PLATE IN COMPRESSION.

buckling stresses for a given loading and material still requires a considerable amount of computation.

The intent of this paper is to present the available data in such a form that the plastic buckling stress can be readily computed for several loading conditions. The resulting charts of the plasticity coefficients produce the desired results and are valid for any material. An example is presented to illustrate the use of the plasticity coefficient charts.

Carpet plots of the plasticity coefficient against the secant and tangent modulus ratios are presented for the following five cases:

(1) A long simply supported plate in compression.

(2) A long simply supported plate under shear.

(3) A simply supported square plate under shear.

(4) A cylinder under external pressure or a simply supported wide plate column.

(5) A cylindrical shell under axial compression.

It should be noted that the data presented are applicable for materials that can be considered isotropic and are restricted to crystalline materials that above the elastic limit behave in a plastic way such as steel, aluminum, magnesium, titanium, etc. The results are applicable to the ultimate strength of the material.

Plastic Buckling

The buckling stress for a plate or shell is usually given by the expression

$$\sigma_{cr} = \{(k\pi^2\eta E)/[12(1 - \mu^2)]\}\ (t/b)^2 \tag{1}$$

where k is the nondimensional buckling coefficient for the configuration under consideration; E is Young's modulus, μ is Poisson's ratio; and η is the plasticity coefficient. The incorporation of the plasticity coefficient, η, makes the above expression applicable in both the elastic and plastic range. In the elastic range, $\eta = 1$, and in the plastic range, η is less than one. The value of η is dependent upon the type of loading, geometry of the plate configuration, and the stress-strain curve in compression.

Eq. (1) as it now stands is not readily applicable to analysis purposes. Since η itself is given as a function of the unknown buckling stress, a trial-and-error calculation is generally required to obtain the required buckling stress.

An alternate procedure, however, eliminates this trial-and-error method. Thus, Eq. (1), by algebraic manipulation, can be put into the form

$$\left(\frac{b}{t}\right)\frac{1}{\sqrt{k}} = \sqrt{\frac{\{(\pi^2E)/[12(1 - \mu^2)]\}}{(\sigma_{cr}/\eta)}} \tag{2}$$

Since the numerator under the radical is a constant for a given material, the equation becomes

$$\left(\frac{b}{t}\right)\frac{1}{\sqrt{k}} = \sqrt{\frac{c}{(\sigma_{cr}/\eta)}} \tag{3}$$

where

$$c = (\pi^2E)/[12(1 - \mu^2)] \tag{4}$$

Eq. (3) can be plotted on log-log paper as $(b/t) \times (1/\sqrt{k})$ vs. σ_{cr} for a specific loading condition and material. To alleviate the routine calculations of η, carpet graphs are utilized which give the values of η in terms of (E_t/E) and (E_s/E). It should be noted that the carpet plots of η are independent of material.

(1) A Long Simply Supported Plate in Compression

The value of η for this case is given by Eq. (53) of reference 1 or from Table 5 of reference 3. Thus,

$$\eta = 0.455(\sqrt{AD} + B + 2F) \tag{5}$$

where the coefficients A, B, D, and F are obtained from Eqs. (21) or (21a) of reference 3, using $\beta = \rho_2/\rho_1 = 0$ or from Eqs. (22) and (23) of reference 4.

The solution of Eq. (5) is given in Fig. 1 for various values of the secant modulus ratio (E_s/E) and tangent modulus ratio (E_t/E).

(2) A Long Simply Supported Plate Under Shear

For this case, the value of η is obtained from Eq. (75) of reference 6.

$$\eta = 0.161[\alpha^{-1}(\{A[A(1 + \alpha^4) + 2(B + 2F)\alpha^2]\}^{1/2} + 3A\alpha^2 + B + 2F)]_{\min} \tag{6}$$

The value of α has to be chosen such as to make the pertinent expression a minimum. A, B, and F are given by Eq. (57) of reference 1 or by the equations on page 345 of reference 5.

The values of η for this case are given in Fig. 2 for various values of the secant and tangent modulus ratios. It should be noted that the compression stress-strain curve may be used for determining these modulus ratios for this shear case. Based on the equivalent uniaxial stress from the distortion energy theory, the associated tangent and secant moduli are obtained for the buckling stress τ_{cr}. From the normal stress

$$\sigma_e = \tau_{cr}\sqrt{3}$$

(3) A Simply Supported Square Plate Under Shear

The value of η which meets the required boundary conditions is given by Eq. (61) of reference 1 or from Table 5 of reference 3. The expression for η is

$$\eta = 6.71(A + B + 2F) \times \sqrt{[41A + 9(B + 2F)]/[8249A + 2601(B + 2F)]} \tag{7}$$

The values of A, B, and F are obtained from Eq. (57) of reference 1 or by the equations on page 345 of reference 5.

Eq. (7) is plotted in Fig. 3. Here again the modulus ratios can be obtained from the compression stress-strain curve. Note that in the case of pure shear

$$\sigma_e = \tau_{cr}\sqrt{3}$$

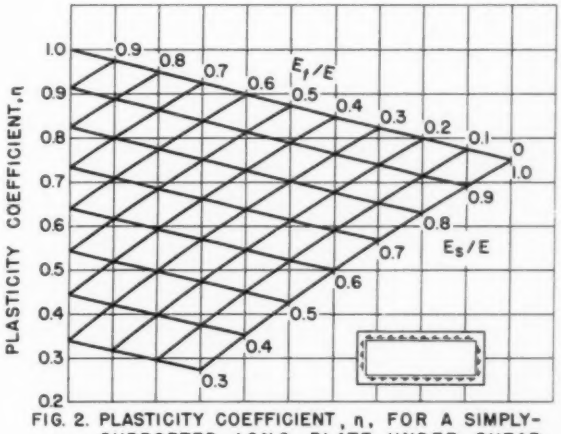


FIG. 2. PLASTICITY COEFFICIENT, η, FOR A SIMPLY-SUPPORTED LONG PLATE UNDER SHEAR.

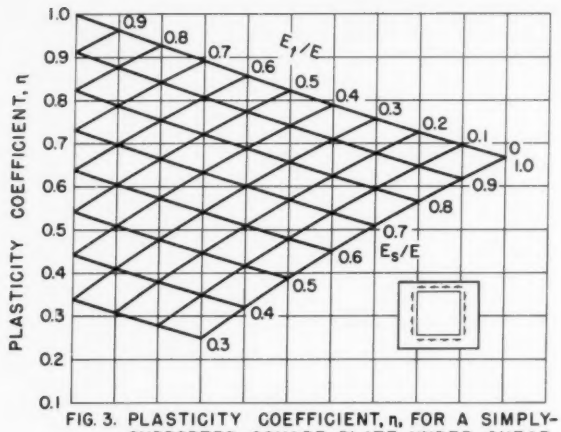


FIG. 3. PLASTICITY COEFFICIENT, η, FOR A SIMPLY-SUPPORTED SQUARE PLATE UNDER SHEAR.

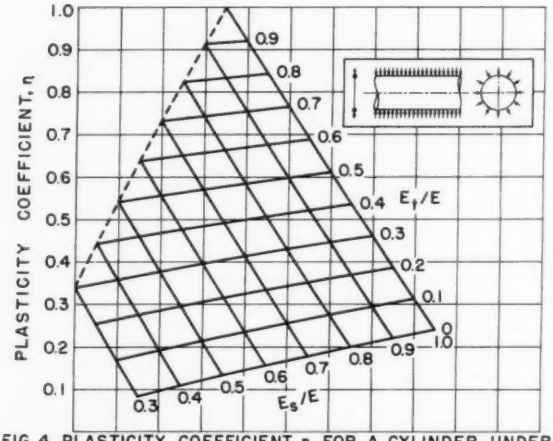


FIG. 4. PLASTICITY COEFFICIENT, η, FOR A CYLINDER UNDER EXTERNAL PRESSURE AND FOR A WIDE PLATE COLUMN.

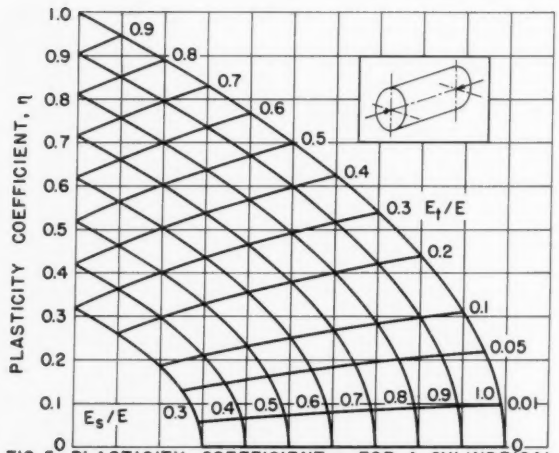


FIG. 5. PLASTICITY COEFFICIENT, η, FOR A CYLINDRICAL SHELL UNDER AXIAL COMPRESSION.

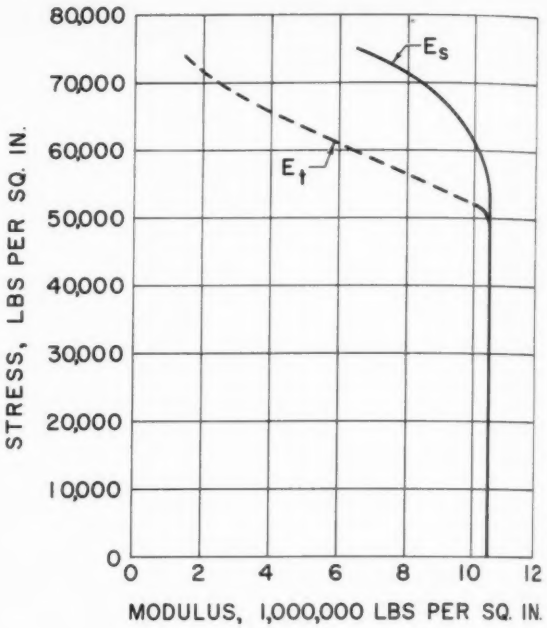


FIG. 6. TANGENT AND SECANT MODULUS 75S-T6 BARE.

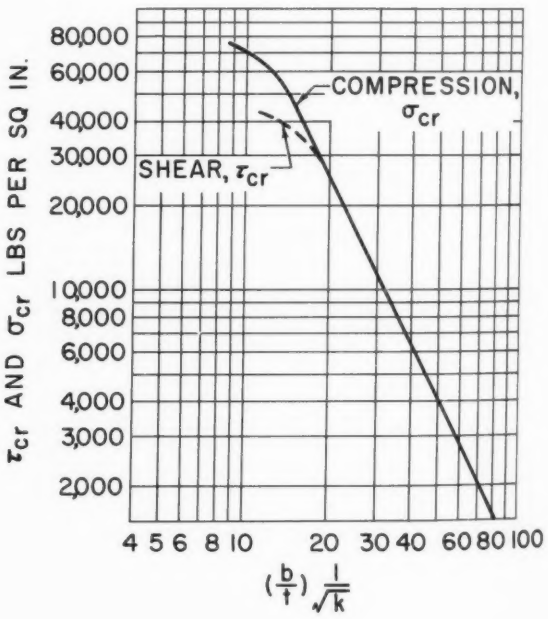


FIG. 7. BUCKLING STRESS FOR LONG SIMPLY-SUPPORTED FLAT PLATE IN COMPRESSION AND UNDER SHEAR—75S-T6 BARE.

TABLE 1
Buckling Stress Computations for a Long Simply Supported Plate in Compression for 75S-T6 Bare Aluminum

(1) σ_{cr}	(2) E_t	(3) E_t/E	(4) E_s	(5) E_s/E	(6) η	(7) σ_{cr}/η	(8) $(\sigma_{cr}/\eta)^{1/2}$	(9) $(b/t)(1/\sqrt{k})$
20,000	10.5×10^6	1.0	10.5×10^6	1.0	1.0	20,000	141.0	22.6
30,000	10.5×10^6	1.0	10.5×10^6	1.0	1.0	30,000	173.2	18.37
40,000	10.5×10^6	1.0	10.5×10^6	1.0	1.0	40,000	200	15.9
50,000	10.45×10^6	0.995	10.5×10^6	1.0	0.999	50,100	224	14.2
54,000	9.10×10^6	0.866	10.48×10^6	0.999	0.984	54,900	234	13.6
58,000	7.32×10^6	0.696	10.31×10^6	0.981	0.946	61,300	248	12.82
62,000	5.60×10^6	0.533	9.92×10^6	0.944	0.895	69,400	264	12.05
66,000	3.90×10^6	0.371	9.31×10^6	0.866	0.822	80,300	284	11.20
70,000	2.48×10^6	0.236	8.40×10^6	0.80	0.730	95,900	310	10.27
74,000	1.50×10^6	0.143	7.0×10^6	0.666	0.605	122,300	350	9.10

Column (6) values are obtained from Fig. 1; Columns (2) and (4) are from Fig. 6.

TABLE 2
Buckling Stress Computations for a Long Simply Supported Plate Under Shear for 75S-T6 Bare Aluminum

(1) σ_{cr}	(2) τ_{cr}	(3) E_t/E	(4) E_s/E	(5) η	(6) τ_{cr}/η	(7) $(\tau_{cr}/\eta)^{1/2}$	(8) $(b/t)(1/\sqrt{k})$
20,000	11,520	1.0	1.0	1.0	11,520	107.2	29.7
30,000	17,300	1.0	1.0	1.0	17,300	131.8	24.2
40,000	23,100	1.0	1.0	1.0	23,100	152.1	20.9
50,000	28,800	0.995	1.0	0.998	28,800	170.0	18.7
54,000	31,100	0.866	0.999	0.965	32,200	179.8	17.7
58,000	33,400	0.696	0.981	0.910	36,700	192.0	16.57
62,000	35,800	0.533	0.944	0.845	42,400	206	15.45
66,000	38,000	0.371	0.866	0.759	50,100	224	14.2
70,000	40,400	0.236	0.80	0.685	59,000	243	13.1
74,000	42,600	0.143	0.666	0.575	74,000	272	11.7

Column (5) values obtained from Fig. 2. Numerical constants for the buckling equation are the same as given in Table 1.

(4) A Cylinder Under External Pressure or a Simply Supported Wide Plate Column

The plasticity coefficient η is given by the second line below Eq. (51) of reference 1.

$$\eta = 0.91A \tag{8}$$

This expression can also be obtained from Table 5 of reference 3. The value of A can be obtained from Eqs. (21) or (21a) of reference 3 using $\beta = \rho_2/\rho_1 = 0$ or Eqs. (22) and (23) of reference 4.

Fig. 4 represents the value of the plasticity coefficient for this case.

(5) A Cylindrical Shell Under Axial Compression

The expression for the plasticity coefficient is given by

$$\eta = \sqrt{0.91(AD - B^2)} \tag{9}$$

This formula is given in Table 5 of reference 3 or taken from Eq. (80) of reference 2. The coefficients A, B, and D are given in Eqs. (21) or (21a) of reference 3 or Eqs. (22) and (23) of reference 4.

This case has been evaluated for the various modulus ratios on the carpet plot of Fig. 5.

ILLUSTRATIVE PROBLEM

The particular example to be considered in detail is designed to illustrate the usefulness of the charts of the plasticity coefficients. The material selected is 75S-T6 bare aluminum, with a compression yield stress of 71,000 lbs. per sq.in. The corresponding tangent and secant moduli for this material are given in Fig. 6.

The following tabular calculations represent the necessary computations for establishing the buckling stress plot. The buckling stresses for a long plate in compression and a long plate under shear, both with simply supported edges, are computed.

$$c = \frac{\pi^2 E}{12(1-\mu^2)} = \frac{(3.14)^2 \times 10.5 \times 10^6}{12(0.91)} = 9{,}480{,}000$$

$$\left(\frac{b}{t}\right)\frac{1}{\sqrt{k}} = \sqrt{\frac{c}{(\sigma_{cr}/\eta)}} = \frac{3{,}180}{\sqrt{\sigma_{cr}/\eta}}$$

The values of the buckling stresses as computed in Tables 1 and 2 are plotted in Fig. 7.

REFERENCES

[1] Bijlaard, P. P., *A Theory of Plastic Stability and Its Application to Thin Plates of Structural Steel*, Proc. of the Royal Netherland Academy of Sciences, Vol. 41, No. 7, pp. 731–743, September, 1938.

[2] Bijlaard, P. P., *Some Contributions to the Theory of Elastic and Plastic Stability*, Publication of the International Association for Bridge and Structural Engineering, Zurich, Vol. 8, pp. 17–80, 1947.

[3] Bijlaard, P. P., *Theory and Tests on the Plastic Stability of Plates and Shells*, Journal of the Aeronautical Sciences, Vol. 16, No. 9, pp. 529–541, September, 1949.

[4] Bijlaard, P. P., *A Theory of Plastic Buckling with Its Application to Geophysics*, Proceedings of the Royal Netherland Academy of Sciences, Vol. XLI, No. 5, pp. 468–480, May, 1938.

[5] Bijlaard, P. P., *Analysis of the Elastic and Plastic Stability of Sandwich Plates by the Method of Split Rigidities—I*, Journal of the Aeronautical Sciences, Vol. 18, No. 5, pp. 339–349, May, 1951.

[6] Bijlaard, P. P., *Analysis of the Elastic and Plastic Stability of Sandwich Plates by the Method of Split Rigidities—II*, Journal of the Aeronautical Sciences, Vol. 18, No. 12, pp. 790–796, December, 1951.

[7] Krivetsky, A., *Plasticity Parameters and Coefficients for Plates and Shells*, Bell Aircraft Corporation Report No. 02-984-019.

Comparison of Mixing Processes in Subsonic Jet Pumps*

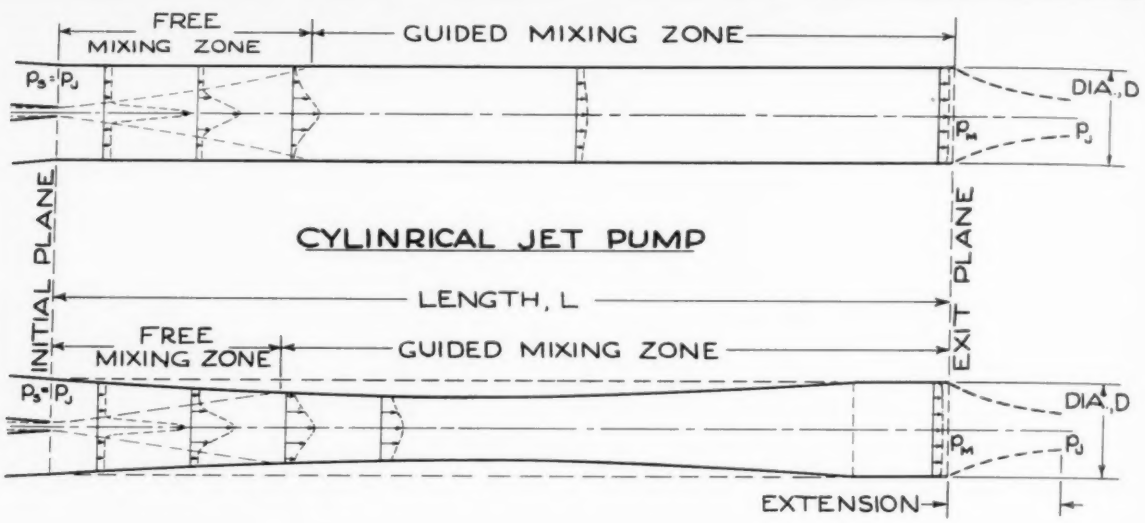
Heinrich B. Helmbold

Staff Engineer, University of Wichita, School of Engineering, Wichita, Kan.

December 16, 1954

ACCORDING TO THE THEORY of reference 1, the ideal efficiency of an axially symmetric jet pump is higher when the jet and secondary flows are mixed at constant static pressure rather than when the flows are mixed in an axially increasing static

* This work was sponsored by the Office of Naval Research under Contract No. N-onr 201(01).



CONSTANT PRESSURE JET PUMP

FIG. 1. Schematic of jet pumps.

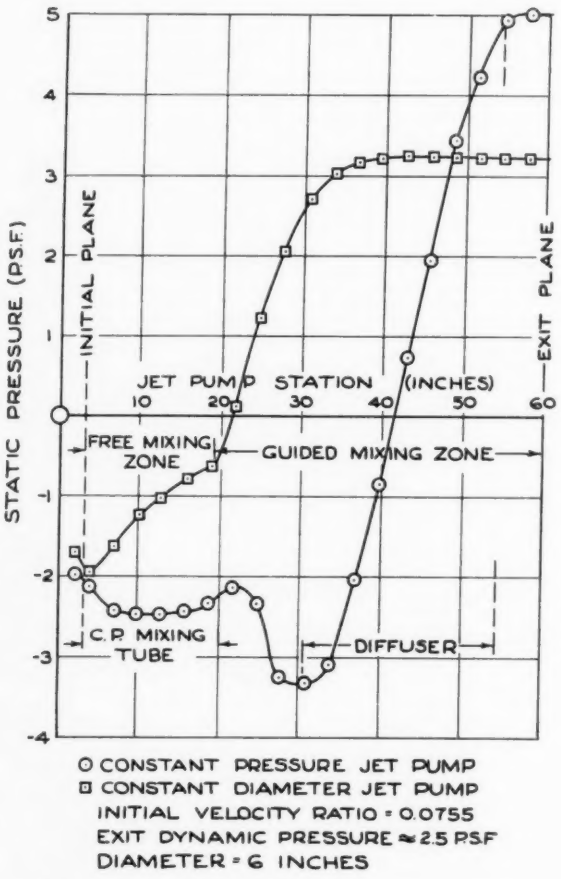


FIG. 2. Static pressure distributions of the jet pumps.

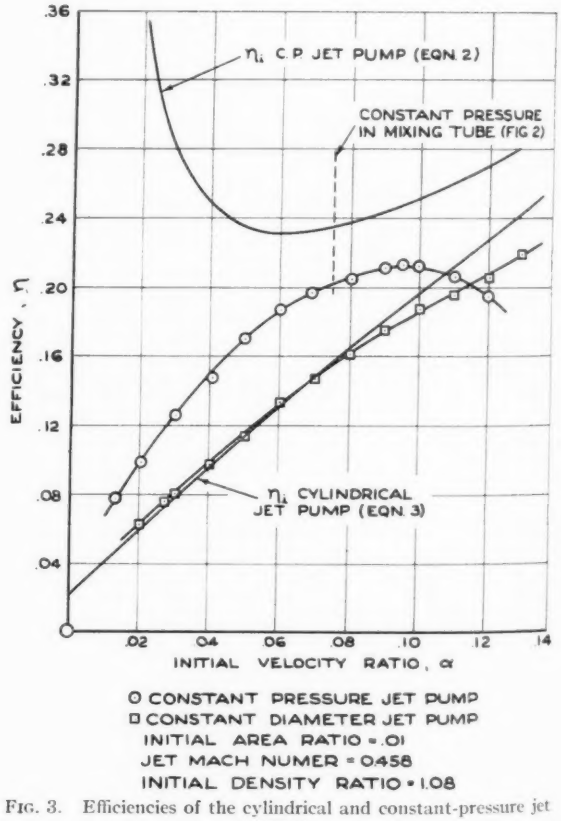

η_i C.P. JET PUMP (EQN. 2)
CONSTANT PRESSURE IN MIXING TUBE (FIG. 2)
EFFICIENCY, η
η_i CYLINDRICAL JET PUMP (EQN. 3)
INITIAL VELOCITY RATIO, α
O CONSTANT PRESSURE JET PUMP
□ CONSTANT DIAMETER JET PUMP
INITIAL AREA RATIO = .01
JET MACH NUMER = 0.458
INITIAL DENSITY RATIO = 1.08

FIG. 3. Efficiencies of the cylindrical and constant-pressure jet pumps.

pressure, as in the case of conventional jet pumps of constant cross-sectional area. More generally, the work performed by the turbulent shearing stresses is decreased when their action is supported by an axially decreasing pressure gradient. Methods of designing constant-pressure mixing tubes were developed in references 3 and 4.

A fair comparison between constant-pressure and constant-area mixing tubes requires that the main dimensions of overall length, initial and exit diameters, and the initial velocities of the jet and secondary flows be identical (see Fig. 1). Thus, the constant-pressure design is handicapped by the contraction of the mixing tube, which requires a diffuser to bring the cross-sectional area back to its initial value. However, it can be predicted that at low initial-velocity ratios the superiority of the constant-pressure design cannot be annihilated by the diffuser losses. With equal initial and exit areas the constant-pressure jet pump will produce a higher static pressure in the exit cross section than the cylindrical jet pump (see Fig. 2). Or, if both the mixed flows are isentropically expanded back to the initial static pressure in an extension of the jet pump (see Fig. 1), the constant-pressure jet pump will produce a greater velocity. This latter condition is used to define a jet-pump efficiency

$$\eta \equiv \frac{m_m}{m_j}\frac{\bar{u}_m^2 - u_s^2}{u_j^2 - u_s^2} = \left(1 + \frac{m_s}{m_j}\right)\left\{\left[\left(\frac{p_{tm}}{p_j}\right)^{(\gamma-1)/\gamma} - 1\right]\frac{\rho_j}{\bar{\rho}_m} - \left[\left(\frac{p_{ts}}{p_j}\right)^{(\gamma-1)/\gamma} - 1\right]\frac{\rho_j}{\rho_s}\right\} \Big/ \left\{\left[\left(\frac{p_{tj}}{p_j}\right)^{(\gamma-1)/\gamma} - 1\right] - \left[\left(\frac{p_{ts}}{p_j}\right)^{(\gamma-1)/\gamma} - 1\right]\frac{\rho_j}{\rho_s}\right\} \quad (1)$$

(m = mass flux, u = velocity, p_t = total pressure, p = static pressure, ρ = mass density, γ = ratio of specific heats; the subscripts are j = jet, s = secondary flow, m = perfectly mixed flow; the bar denotes the mixed flow condition at initial static pressure $p_j = p_s$). The ideal efficiency of the *constant-pressure mixing tube* is simply

$$\eta_i = [m_j u_j + (m_j + 2m_s)u_s]/[(m_j + m_s)(u_j + u_s)] \quad (2)$$

The ideal efficiency of the *cylindrical mixing tube* is

$$\eta_i = [2(m_j u_j + m_s u_s)u_m - (m_j + m_s)(u_s^2 + u_m^2)]/[m_j(u_j^2 - u_s^2)] \quad (3)$$

if

$$(p_m/p_j) - 1 = \gamma M_j^2 (A_j/A_m)\left\{[m_j u_j + m_s u_s - (m_j + m_s)u_m]/(m_j u_j)\right\} \ll 1$$

(M = Mach Number, A = cross-sectional area.) The velocity u_m of the mixed flow in the exit cross section is represented by the equation

$$\frac{u_m}{u_j} = \left[1 + \frac{A_s}{A_j} + \gamma M_j^2\left(1 + \frac{m_s u_s}{m_j u_j}\right)\right] \div \left[1 + \frac{\rho_0}{\rho_1}\frac{m_s}{m_j} + \frac{\gamma - 1}{2}M_j^2\left(1 + \frac{m_s u_s^2}{m_j u_j^2}\right)\right] \quad (4)$$

if

$$2(\gamma + 1)M_j^2\left(1 + \frac{m_s}{m_j}\right)\left\{\left[1 + \frac{A_s u_s}{A_j u_j} + \frac{\gamma - 1}{2}M_j^2\left(1 + \frac{m_s u_s^2}{m_j u_j^2}\right)\right] \Big/ \left[1 + \frac{A_s}{A_j} + \gamma M_j^2\left(1 + \frac{m_s u_s}{m_j u_j}\right)\right]^2\right\} \ll 1$$

The more complex exact formulas are given in reference 5.

With incompressible flow ($M_j = 0$) and equal initial densities ($\rho_j = \rho_s$), the ratio of the work of turbulent shearing stresses in the cylindrical mixing tube to the work in the constant-pressure tube simplifies to

$$\frac{1 - \eta_i\ (\text{cyl.})}{1 - \eta_i\ (\text{c.p.})} = 1 + \frac{A_j}{A_j + A_s}\frac{(u_j - u_s)^2}{u_j u_s} \geq 1 \quad (5)$$

An experimental comparison has been made between a cylindrical jet pump and a constant-pressure jet pump designed for an initial velocity ratio of $u_s/u_j = 0.069$ (reference 6). The initial cross-sectional area ratio of both jet pumps was $A_j/A_s = 0.01$, and their length/diameter ratio was $L/D = 10$. The experimental data are shown in Figs. 2 and 3, while experimental velocity profiles are shown in Fig. 1.

The constant-pressure jet pump. At the design velocity ratio, 81 per cent of the excess kinetic energy in the jet should have been transmitted to the secondary air under constant pressure in the free mixing zone. Since the spreading angle of the jet in still air was greater than that assumed in the design, the experiments showed approximately constant mixing pressure at the somewhat higher velocity ratio of $u_s/u_j = 0.0755$ (see Fig. 2). The experimental efficiency reached 87 per cent of the ideal efficiency at this velocity ratio (see Fig. 3). At lower velocity ratios the efficiency suffered from mixing under adverse (positive) pressure gradients, while at higher velocity ratios the efficiency first improved by mixing under more favorable (negative) pressure gradients, but was then again reduced by increasing diffuser losses and finally by incipient instability of the imperfectly mixed diffuser flow. The superiority of the constant-pressure design over the conventional jet pump was evident over a wide range of working conditions; it amounted to a 31 per cent improvement with constant pressure in the mixing tube.

The cylindrical jet pump. The additional losses in the cylindrical jet pump are essentially frictional losses in the boundary layer on the tube wall. Therefore, they were negligible at low velocity ratios. By assuming potential flow outside of the mixing zone, it was predicted in reference 2 that reverse flow would occur in the cylindrical mixing tube when the total-pressure coefficient $[c_p \equiv 2(p_{tm} - p_{ts})/\rho_m u_m^2]$ exceeded 4.556. The experiments showed that under the influence of the boundary layer, reverse flow occurred for total-pressure coefficients greater than 2.76—i.e., at velocity ratios lower than $u_s/u_j = 0.0696$. A ring vortex that oppressed the adverse pressure gradient was formed by the flow reversal in a downstream section of the free mixing zone. This raised the experimental mixing efficiency slightly above its ideal value (see Fig. 3).

References

[1] Helmbold, H. B., *Contributions to Jet Pump Theory, I. Comparison of Ideal Mixing Processes*, University of Wichita Engineering Study No. 105, June, 1953.

[2] Helmbold, H. B., *Contributions to Jet Pump Theory, II. Integral Relations on Mixing Processes*, University of Wichita Engineering Study No. 106, June, 1953.

[3] Helmbold, H. B., *Contributions to Jet Pump Theory, III. Simplified Theory of Mixing-Zone Spreading*, University of Wichita Engineering Study No. 107, July, 1953.

[4] Helmbold, H. B., *Contributions to Jet Pump Theory, IV. Approximate Theory of Jet Diffusion in a Constant-Pressure Mixing Tube*, University of Wichita Engineering Study No. 118, November, 1953.

[5] Helmbold, H. B., *Contributions to Jet Pump Theory, VI. Perfect Mixing Processes in Subsonic Jet Pumps*, University of Wichita Engineering Study No. 145, June, 1954.

[6] Helmbold, H. B., Luessen, G., and Heinrich, A., *Experimental Comparison of Constant-Pressure and Constant-Area Jet Pumps*, University of Wichita Engineering Report No. 147, July, 1954.

A Simplified Treatment of Partial Fractions for Dynamic-Response Calculations

George S. Campbell
California Institute of Technology, Pasadena, Calif.

December 23, 1954

IN CALCULATING THE DYNAMIC RESPONSE of a linear system, the numerical evaluation of partial-fraction coefficients corresponding to complex characteristic roots is usually a laborious operation. The present note demonstrates a method by which,

in certain instances, simple exact expressions for these complex coefficients may be derived. The method is particularly suited to cases in which there is only one pair of complex roots, with no limitation on the number of real roots to the characteristic equation. For more than one pair of complex roots, only "the last" pair may be evaluated by the present approach, the others being obtained by the usual residue methods.

In order to demonstrate the method of approach, it will be convenient to consider the lateral response of aircraft to step control inputs. The specific results obtained are applicable whenever the stability quartic has two real roots and a pair of conjugate-complex roots. Notation identical to that of reference 1 is used throughout.

Illustration of Method

Consider the Laplace transform of the bank angle ϕ, as given in reference 1,

$$\phi_\lambda = \frac{a_0\lambda^5 + a_1\lambda^4 + a_2\lambda^3 + a_3\lambda^2 + a_4\lambda + a_5}{\lambda^2(A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E)} \tag{1}$$

This expression may also be written in terms of the factors of the stability quartic

$$\phi_\lambda = \frac{u_\lambda}{v_\lambda} = \frac{a_0\lambda^5 + a_1\lambda^4 + a_2\lambda^3 + a_3\lambda^2 + a_4\lambda + a_5}{A\lambda^2(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)} \tag{2}$$

If the roots of the stability quartic are distinct and non-zero, the partial-fraction expansion of ϕ_λ is

$$\phi_\lambda = \frac{A_1}{\lambda - \lambda_1} + \frac{A_2}{\lambda - \lambda_2} + \frac{A_3}{\lambda - \lambda_3} + \frac{A_4}{\lambda - \lambda_4} + \frac{A_5}{\lambda^2} + \frac{A_6}{\lambda} \tag{3}$$

In the analysis to follow, expressions for the real and imaginary parts of A_1 and A_2 are to be derived. The process is essentially one of rearranging Eqs. (2) and (3) so that the unknown complex coefficients are expressed in terms of the real coefficients A_3, A_4, . . . which are easily calculated by applying the usual residue rules. [See Eqs. (6) of reference 1.]

The complex roots λ_1 and λ_2 are denoted by

$$\left.\begin{aligned}\lambda_1 &= R + Ii\\ \lambda_2 &= R - Ii\end{aligned}\right\} \tag{4}$$

and the complex response coefficients A_1 and A_2 are written as

$$\left.\begin{aligned}A_1 &= R_A + I_A i\\ A_2 &= R_A - I_A i\end{aligned}\right\} \tag{5}$$

With this notation, Eq. (3) takes the form

$$\phi_\lambda = \frac{2R_A(\lambda - R) - 2I_A I}{Q} + \frac{A_3}{\lambda - \lambda_3} + \frac{A_4}{\lambda - \lambda_4} + \frac{A_5}{\lambda^2} + \frac{A_6}{\lambda} \tag{6}$$

where

$$Q = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - 2R\lambda + (R^2 + I^2) \tag{7}$$

If the equivalent expressions [Eqs. (2) and (6)] are set equal to each other, one finds that

$$2R_A(\lambda - R) - 2I_A I = -\frac{A_6 Q}{\lambda} - \frac{A_5 Q}{\lambda^2} + \frac{u_\lambda - A\lambda^2 Q[(A_3 + A_4)\lambda - (\lambda_4 A_3 + \lambda_3 A_4)]}{A\lambda^2(\lambda - \lambda_3)(\lambda - \lambda_4)} \tag{8}$$

Because the left side of Eq. (8) is linear in λ, the sum of the terms on the right must also be linear in λ. Therefore, only the terms of order zero and one from each fraction enter into the present calculation, the other terms canceling identically. Making use of this observation, Eq. (8) may be written as

$$2R_A(\lambda - R) - 2I_A I = -A_6(\lambda - R) + RA_6 - A_5 + [(a_0/A) - A_3 - A_4](\lambda - R) + (R + \lambda_3 + \lambda_4)(a_0/A) + (a_1/A) + R(A_3 + A_4) - \lambda_3 A_3 - \lambda_4 A_4 \tag{9}$$

Equating the coefficients of $(\lambda - R)$ on the two sides of Eq. (9) provides the real part of the unknown coefficient A_1:

$$2R_A = (a_0/A) - (A_3 + A_4 + A_6) \tag{10}$$

Equating the constant terms of Eq. (9) provides

$$2II_A = (Ba_0/A^2) - (a_1/A) + 2RR_A + \lambda_3 A_3 + \lambda_4 A_4 + A_5 \tag{11}$$

The particular form in which I_A is expressed results from use of the fact that the sum of the roots of the stability quartic equals $(-B/A)$.

Summary of Results

For those interested in the specific results obtained for the lateral response of aircraft to step control inputs, expressions for the real and imaginary parts of the complex response coefficients for roll, yaw, and sideslip are summarized below. It is assumed that the stability quartic has two real roots and a pair of conjugate complex roots. These results are expressed in the notation of reference 1 and are directly applicable to the problem considered there.

$$2R_A = (a_0/A) - (A_3 + A_4 + A_6)$$

$$2II_A = (Ba_0/A^2) - (a_1/A) + 2RR_A + \lambda_3 A_3 + \lambda_4 A_4 + A_5$$

$$2R_B = (b_0/A) - (B_3 + B_4 + B_6)$$

$$2II_B = (Bb_0/A^2) - (b_1/A) + 2RR_B + \lambda_3 B_3 + \lambda_4 B_4 + B_5$$

$$2R_C = (c_0/A) - (C_3 + C_4 + C_5)$$

$$2II_C = (Bc_0/A^2) - (c_1/A) + 2RR_C + \lambda_3 C_3 + \lambda_4 C_4$$

Reference

[1] Campbell, John P., and McKinney, Marion O., *Summary of Methods for Calculating Dynamic Lateral Stability and Response and for Estimating Lateral Stability Derivatives*. NACA Report No. 1098, 1952 (formerly NACA TN 2409).

On the Possibility of Aeroelastic Reversal of Propeller Blades*

F. W. Niedenfuhr
Engineering Mechanics Department, Ohio State University, Columbus, Ohio
December 30, 1954

It is the purpose of this note to call attention to an elastic instability effect that may occur on thin pretwisted propeller blades or other similar sections.

The relation between torsional moment, M_T, and angle of twist per unit length for a thin flat section with no pretwist is

$$M_T = k_1\varphi + k_2\varphi^3 \tag{1}$$

Timoshenko[1] gives the constants k_1 and k_2 as $k_1 = bt^3G/3$ and $k_2 = Ebt^5/360$ for a narrow rectangular cross section of width b and thickness t.

Stone and Schulze[2] have shown that when torque is applied to a slab that has been pretwisted an amount ψ per unit length, the stiffness of the slab is found to be increased so that the governing equation is

$$M_T = k_1\varphi + k_2[(\psi + \varphi)^3 - \psi^3(\psi - \varphi)] \tag{2}$$

It is interesting to look at this equation in graphical form. Fig. 1 shows a plot for one set of values of k_1, k_2 and ψ, and Fig. 2 shows the plot for different values of these parameters.

Here, we have used the convention that φ is positive when it is in the direction of the pretwist. Clearly, if a propeller blade has the characteristics shown by Fig. 2, the twist angle will take a sudden jump at some torque while negative torque is being applied (lower dotted line). Similarly, on unloading, the twist will suddenly reduce at constant torque (upper dotted line).

If the propeller blade is to be statically stable the torque-twist relation should always be as shown in Fig. 1—i.e., it should have

* This paper is based on part of the work done by the author under WADC Contract No. AF33(616)2124 (Ohio State University Research Foundation Project No. 569).

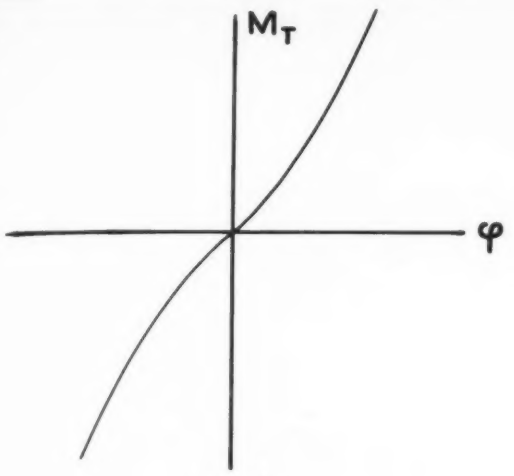


FIG. 1.

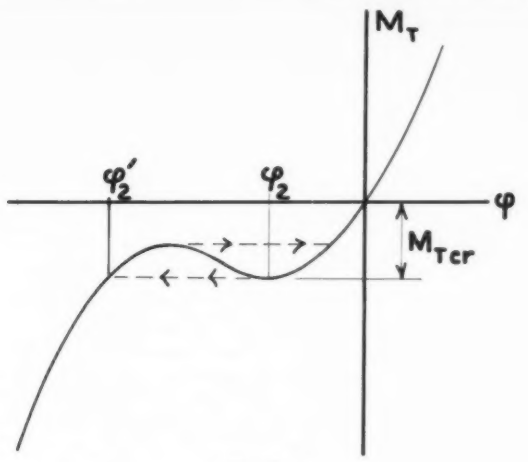


FIG. 2.

positive slope everywhere. This condition can be met by placing restrictions on the thickness ratios and pretwist angles used. The analysis is as follows:

The slope of the torque vs. twist curve is

$$\partial M_T/\partial\varphi = 3C_2Etb^3\varphi^2 + 6C_2Etb^3\psi\varphi + (C_1Gt^3b + 2C_2Etb^3\psi^2) \quad (3)$$

where C_1 and C_2 are defined by the equations $k_1 = C_1Gt^3b$ and $k_2 = C_2Etb^3$. Symbolically, $\partial M_T/\partial\varphi = A\varphi^2 + B\varphi + C$. This relation is shown in Fig. 3. The zeros of the quadratic form are φ_1 and φ_2, given by

$$\varphi_{1,2} = (-B \pm \sqrt{B^2 - 4AC})/2A$$

It can be easily seen that $\varphi_1 \leqq -\psi \leqq \varphi_2 < 0$, so that if $\partial M_T/\partial\varphi$ is negative the abrupt change of twist angle at constant torque occurs before the blade reaches zero total twist on applying or releasing negative moments. This means that a propeller blade might suddenly reverse its total pitch when the aerodynamic torques reach a certain value. (The sum of the torques on a rotating propeller blade tends to take out the pretwist.)

$\partial M_T/\partial\varphi$ will be greater than or equal to zero everywhere provided that $B^2 - 4AC \leqq 0$, for then φ_1 and φ_2 will be either equal reals or complex conjugates, and the curve of Fig. 3 will be wholly above the φ axis. From Eq. (3), then, we have on setting $B^2 - 4AC \leqq 0$ and simplifying:

$$t^2/b^4 \geqq \psi^2C_2E/C_1G \quad (4)$$

The constant C_1 can be calculated by the membrane analogy for torsion, and the constant C_2 has been calculated for various airfoil cross sections.[2] For symmetrical NACA 16 series airfoils $C_1 = 0.178$ and $C_2 = 0.001245$. For symmetrical diamond cross sections $C_1 = 0.0833$ and $C_2 = 0.0006076$.

Denoting the percentage thickness ratio of the blade by r, Eq. (4) becomes

$$r/b \geqq C_3\psi \quad (5)$$

where the constant $C_3 = 10^2\sqrt{(C_2E)/(C_1G)}$ is shown for some special cases in Table 1. Note that the improvement resulting from the use of steel instead of aluminum alloys is only of the order of 3 per cent since the ratio E/G is nearly the same for both materials.

TABLE 1
Values of C_3 for Eq. (5)

	NACA 16 series	Diamond shape
Aluminum alloy	13.61	13.90
Steel	13.22	13.50

Fig. 4 shows a plot of Eq. (5) for a symmetrical NACA 16 series airfoil of aluminum alloy. As an example, a propeller blade of 15-in. chord with a pretwist angle of 0.010 radians per in. must be thicker than 2 per cent in order not to exhibit the reversal tendency.

The moments, stresses, and twist angle changes involved in the instability effect can be calculated and shed more light on the phenomenon. The angle, φ_2, at which a twisted propeller blade "jumps" is obtained from the quadratic formula. After simplification it becomes

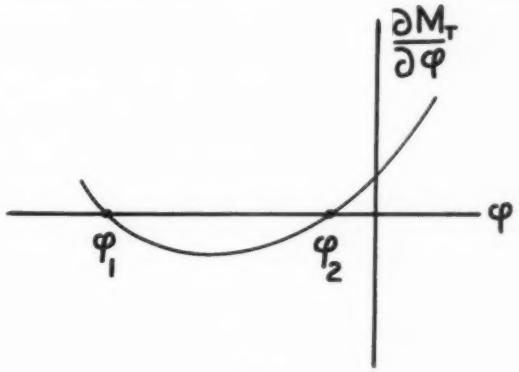


FIG. 3.

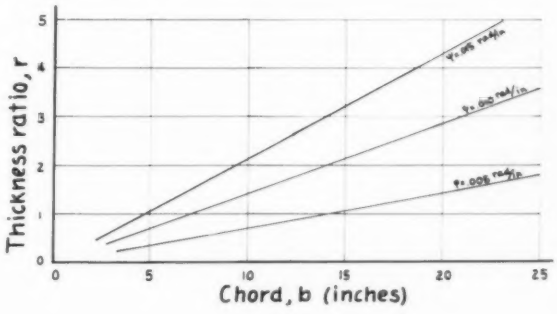


FIG. 4.

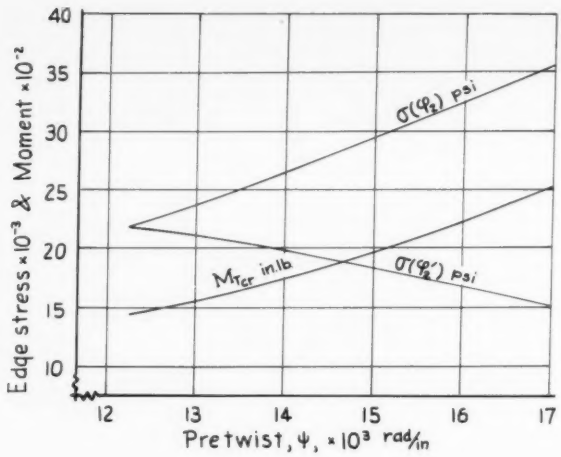


FIG. 5.

$$\varphi_2 = \psi\left(-1 + \sqrt{(1/3)\left\{1 - [(C_1 G t^2)/(C_2 E b^4 \psi^2)]\right\}}\right) \quad (6)$$

For a given blade, this critical angle can be calculated, and then the critical torque can be obtained from Eq. (2). Denoting the critical torque by $M_{T_{cr}}$ and setting the right-hand side of Eq. (2) equal to $M_{T_{cr}}$, the angle φ_2' can be solved for, where φ_2' is the twist after the sudden jump.

Reference 1 discusses the distribution of tensile and compressive stresses over the cross section of a thin twisted slab. It is shown there that a tensile stress of considerable magnitude is developed at the edges. Reference 2 develops the discussion more fully and shows that there is a longitudinal stress at the leading and trailing edges of a propeller blade given by Eq. (7). The constant 0.0963 is for symmetrical NACA 16 series airfoils.

$$\sigma = 0.0963 E b^2 [(\psi + \varphi)^2 - \psi^2] \quad (7)$$

Substituting φ_2 and φ_2' into Eq. (7), the edge stress before and after the jump can be calculated. Since φ_2 is negative and less than ψ, the edge stress is compressive before the jump. Furthermore, it will be seen that when the twist is equal to φ_2' the edge stress is either a smaller compressive stress than at φ_2 or else a tensile stress. The reason for this is that the jump of the blade from φ_2 to φ_2' is really a sudden release of potential energy.

Fig. 5 shows the critical twisting moment and edge stresses in the blade before and after the jump for a blade of symmetrical NACA 16 series airfoil of aluminum alloy with 12-in. chord and 2 per cent thickness. The calculations for this figure are from Eqs. (2) and (7).

It should be noted that Eq. (5) is not an entirely satisfactory criterion for design purposes, since it merely states that the the torque-twist curve shall not have negative slope. Even propeller blades that satisfy the inequality in Eq. (5) may exhibit dangerous tendencies, however, since there is a wide range of rather moderate pretwist angles which give the torque-twist curve an extremely small slope, so that extremely large increments of deflection are obtained with small increments of load. Also, in the region where the slope of the curve is small, Eq. (2) is sensitive to small changes of parameters. Calculations on a blade of 16-in. chord and 2 per cent thickness show that 2 or 3 per cent errors in the actual thicknesses lead to errors of the predicted twist of the order of 50 per cent. Similar changes of predicted twist for a given moment are introduced by small inaccuracies in the assumed pretwist. Experiments with a 16-in. chord 2 per cent thick blade of aluminum alloy bear out this extreme sensitivity to minor changes of the parameters.

REFERENCES

[1] Timoshenko, S., *Strength of Materials*, Part II, pp. 298–304, D. Van Nostrand Company, Inc., New York, 1947.

[2] Stone, F. R., and Schulze, J. O., *Calculation of Steady Longitudinal Stresses Arising from Torsional Deflection of a Thin Solid Pretwisted Propeller Blade Segment without and with Rotation*, WADC Technical Report 53-20, March, 1933.

Heat Conducting Bodies in Boundary-Layer Flow*

Salomon Levy
Guided Missiles Department, General Electric Company, Schenectady, N.Y.

January 3, 1955

SYMBOLS

A	=	cross-sectional area at distance x
h	=	heat-transfer coefficient at distance x
h_L	=	heat-transfer coefficient at distance L
k	=	thermal conductivity
K	=	kernel function of integral equation
L	=	distance from tip or leading edge to base
p	=	temperature gradient
Q	=	heat absorbed by sink
r	=	heat generated or irradiated
s	=	surface area from tip to distance x
T	=	body temperature at distance x
T_{iw}	=	insulated wall or recovery temperature
T_L	=	temperature at distance L
T_0	=	tip or leading-edge temperature
w	=	temperature difference, $T - T_{iw}$
x	=	distance from tip or leading edge
z	=	variable introduced in Eq (4)

DISCUSSION

IN A RECENT NOTE[1] Denison presented values for the tip temperature of a cone or wedge in laminar high-speed flight and based his results upon solutions of the differential equation

$$(d/dx)[kA(dw/dx)] - h(ds/dx)w = 0 \quad (1)$$

where

$$w = T_{iw} - T \quad (2)$$

$$h = h_L(x/L)^{-1/2} \quad (3)$$

The form of Eq. (3) suggests a constant wall temperature, and its use may be incompatible with the large temperature gradient anticipated at the tip. The heat-transfer coefficient h depends upon the wall temperature variation, and a more correct expression is given in reference 2—namely,

$$\int_0^x h(x, z)\, dT(z) = -\int_0^x h(x, z)\, dw(z) \quad (4)$$

The functions $h(x, z)$ have been tabulated by Tribus and Klein[2] for boundary-layer or pipe flow of a laminar or turbulent nature.

In more general terms Eq. (1) can be rewritten

$$(d/dx)[kA(dw/dx)] + (ds/dx)\int_0^x h(x, z)\,[dw(z)/dz]\, dz = -r(x) \quad (5)$$

where $r(x)$ represents the heat generated within or irradiated upon a unit area of body surface.

Replacement of dw/dx by $p(x)$ and integration with respect to x between 0 and x give

$$(kAp)_x - (kAp)_0 + \int_0^x \frac{ds}{dx}\, dx \int_0^x h(x, z)p(z)\, dz = -\int_0^x r(x)\, dx \quad (6)$$

* The majority of this material was presented as an oral discussion of ASME Paper No. 53-SA-46 at the Semi-Annual Meeting of the ASME at Los Angeles, Calif., June, 1953. Most of this work was carried out at the Boundary Layer Project of the University of California at Berkeley.

Use of the Dirichlet's theorem to invert the order of integration in the second term yields

$$(kAp)_x - (kAp)_0 + \int_0^x K(x, z)\, p\,(z)\, dz = R(x) \qquad (7)$$

where

$$R(x) = -\int_0^x r(x)\, dx$$

$$K(x, z) = \int_0^x (ds/dx) h(x, z)\, dx$$

Eq. (7) is a Volterra integral equation of the second kind, and it specifies the temperature gradient at the surface. For the boundary condition utilized by Denison[1]

$$(kAp)_{x=L} = Q \text{ and } R(x) = 0$$

Eq. (7) becomes

$$Q - kAp + \int_x^L K(x, z)\, p\,(z)\, dz = 0 \qquad (8)$$

Eq. (8) can be solved for the gradient p in terms of Q. The method of finite difference presented by Fox and Goodwin[3] appears most suited in this instance. Once the gradient p is determined, one additional quadrature yields the excess temperature $(T - T_L)$ in terms of distance from the tip. If $T_L = T_{tw}$, the tip temperature excess $T_0 - T_{tw} = T_0 - T_L$ is obtained for $x = 0$.

Eq. (7) is general and applies to laminar or turbulent boundary layer and pipe flow. When the body is radiating the function $r(x)$ varies as T^4 and Eq. (7) becomes a nonlinear integral equation. Then the solution obtains by successive approximations. For the particular problem treated by Denison the function $h(x, z)$ may be written

$$h(x, z) = f(x)\,(x^m - z^m)^{-n}$$

where m and n are prescribed constants.[2]

References

[1] Denison, R. M., *Tip or Leading-Edge Temperatures on Pointed Heat Conducting Bodies at High Supersonic Speeds*, Journal of the Aeronautical Sciences, Readers' Forum, Vol. 21, No. 12, p. 858, December, 1954.

[2] Tribus, M., and Klein, J., *Forced Convection from Nonisothermal Surfaces*, Heat Transfer Symposium, p. 211, Engineering Research Institute, University of Michigan, 1953. Also ASME Paper No. 53-SA-46.

[3] Fox, L., and Goodwin, E. T., *The Numerical Solution of Non-Singular Linear Integral Equations*, Philosophical Transactions of the Royal Society of London, Series A, No. 902, Vol. 245, February, 1953.

On the Calibration of Supersonic Wind Tunnels

Jacques A. F. Hill
Research Engineer, Naval Supersonic Laboratory, MIT

January 8, 1955

SOME DIFFICULTIES ENCOUNTERED in the calibration of a supersonic wind-tunnel nozzle have recently been reported by Murphy.[1] From measurements of static and pitot pressure, the Mach Number was determined in three ways:

(1) From the ratio of static pressure p_S to stilling-section pressure p_T, assuming isentropic flow in the nozzle.

(2) From the ratio of pitot pressure p_P to p_T, assuming isentropic flow in the nozzle.

(3) From the ratio of p_S to p_P, without this assumption.

By extrapolating the results to a low dew point ($-40°$F.), the following Mach Numbers were obtained at a particular point in an $M = 2.84$ nozzle: (1) 2.851, (2) 2.877, and (3) 2.818. Similar discrepancies have been observed elsewhere and have sometimes been ascribed to a stagnation-pressure loss in the expansion through the nozzle. Murphy shows that a loss of the required magnitude cannot be accounted for theoretically by any method so far suggested.

Other explanations may be based on inherent errors in the measurement of either pitot or static pressure. The effect of viscous forces on the measurement of pitot pressure has been investigated by Sherman,[2] using various probe shapes. From his experimental results one may conclude that the effect is negligible (representing <0.1 per cent error) for $Re > 10^3$, based on free-stream conditions. Murphy's probe and those used in this investigation were open-ended circular tubes run at $Re > 10^4$. One might suspect the existence of another type of error for such probes, associated with the finite hole size and increasing with the ratio of I.D. to O.D. However, the four probes shown in Fig. 1 gave the same reading (within the measurement accuracy of about ±0.05 per cent) at $M = 3.5$. Any inherent error in measurements with pitot probe A at atmospheric or greater stagnation pressures in a supersonic wind tunnel should, therefore, be extremely small.

The measurement of static pressure, on the other hand, is affected by the displacement effect of the boundary layer on the probe. An approximate theoretical estimate of this effect for the probe of reference 1 gives a static-pressure error of almost 3 per cent; whereas an error of about 4 per cent would closely account for the discrepancies noted above.

Pitot pressure thus seems to be the pressure that may be most accurately measured in a supersonic flow. To calibrate a wind tunnel independently of the assumption in (2) above, another measurement is necessary in addition to the pitot pressure in the undisturbed stream. It is the purpose of this note to recommend the measurement of pitot pressure, p_{PS}, behind a known oblique shock. This pressure is, of course, a function of the shock angle ω. At first sight one would expect the accuracy of the calibration to be severely limited by the difficulties in the accurate measurement of shock angles in shadowgraphs. The pressure in question, however, regarded as a function of ω with other conditions fixed, exhibits a maximum for some angle ω^* close to that which gives equal values of specific entropy rise across the oblique and pitot shocks. Near this point the variation of p_{PS} with ω is extremely slow.

In Fig. 2 the optimum shock angle ω^* and the corresponding flow deflection angle δ^* are shown as a function of Mach Number. In Figs. 3 and 4 typical curves for calibrating $M = 3.5$ and $M = 7$ nozzles are shown. The ordinate is the ratio of the two pitot pressures $P = p_{PS}/p_P$. Curves are drawn for shock angles equal to ω^* and $^1/_2$ degree either side of this value. Clearly a crude measurement of shock angle is sufficient, and the accuracy of calibration is limited only by the accuracy in the measurements of the pressures. At $M = 3.5$ the percentage error in M is about equal to and at $M = 7$ it is about twice that in P.

In March, 1954, a calibration by this method was carried out for one point in the center of the test section of the $M = 3.5$ nozzle in the MIT Naval Supersonic Laboratory tunnel. Now in the computations for the curves of Figs. 3 and 4 it has been assumed that the Mach Number upstream of the pitot shock is equal to that just downstream of the oblique shock. In any practical configuration these two shocks are separated by a distance d of the order of a few tenths of an inch. Over this distance the flow is influenced by expansion and compression waves from the boundary layer on the wedge and from the nozzle walls, the latter being associated with the nonuniform flow in the empty tunnel. An approximation to the value of p_{PS} corresponding to the theoretical case was obtained by extrapolating measurements for several values of d to a value for $d = 0$. The results are shown in Fig. 3; together with estimates of the inaccuracies in the measurements they are

$$\omega = 39° \pm 0.1°$$

$$p_P/p_T = 0.2073 \pm 0.0001$$

$$p_{PS}/p_T = 0.4234 \pm 0.0001$$

$$P = 2.042(5) \pm 0.0015$$

$$M = 3.529 \pm 0.003$$

$$p_0/p_T = 0.998 \pm 0.003$$

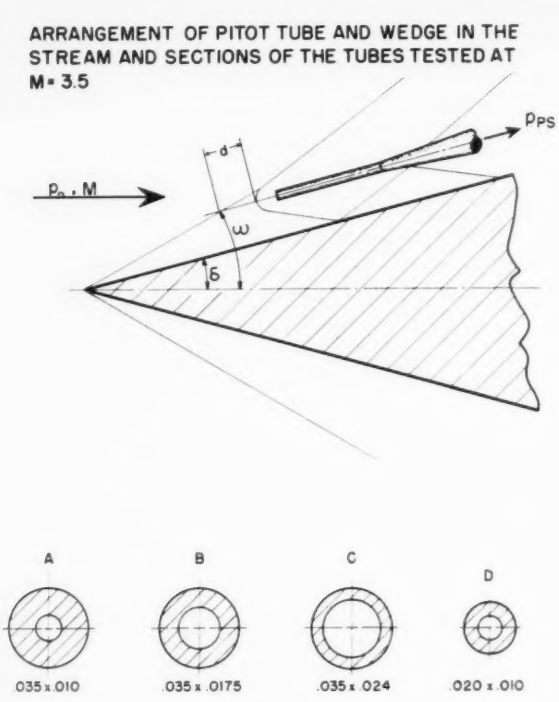


FIG. 1.

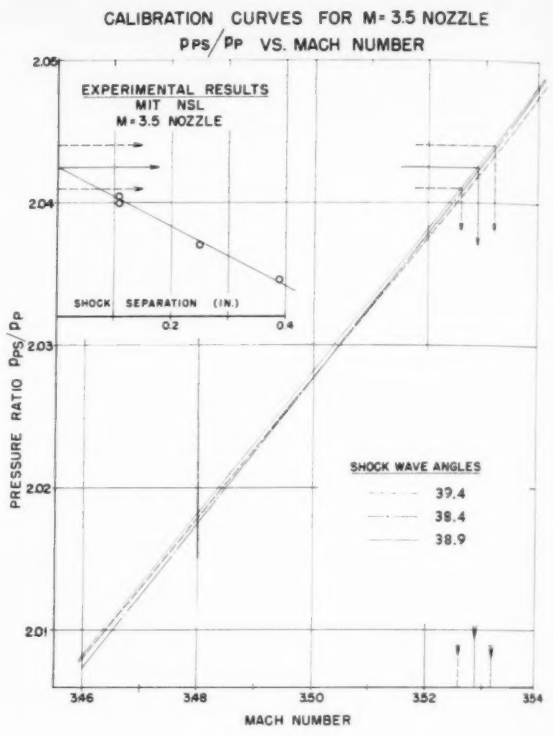


FIG. 3.

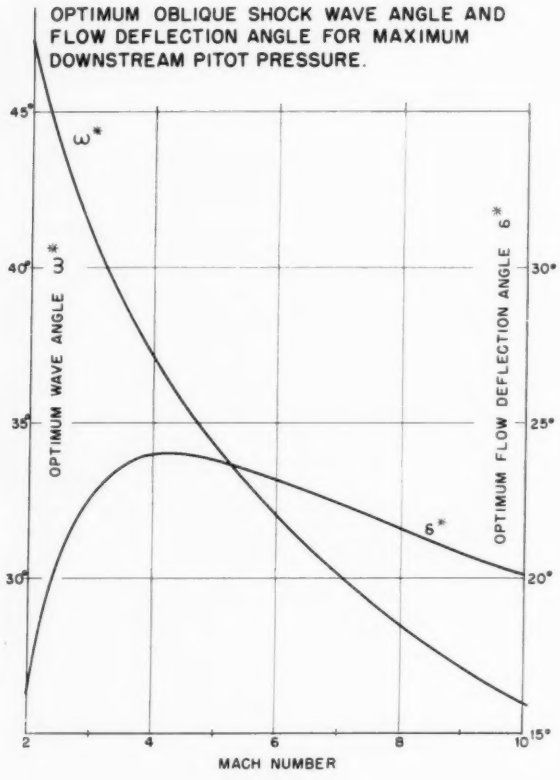


FIG. 2.

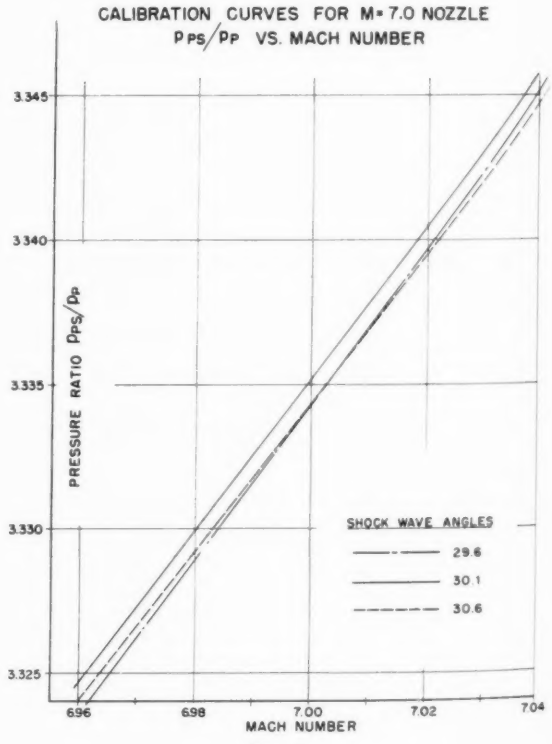


FIG. 4.

where p_0 represents the stagnation pressure in the test section. These measurements were carried out at a dew point of about $-50°$F. A theoretical estimate of the stagnation-pressure loss associated with condensation gives

$$p_0/p_T = 0.999$$

It is usually found, however, that pitot-pressure measurements do not fully indicate condensation losses.

On the basis of these results one may conclude that the expansions in supersonic wind-tunnel nozzles are generally isentropic (in the absence of condensation) and that the most reliable method of calibration is (2) above. A duplication of this work in a hypersonic nozzle seems desirable.

References

[1] Murphy, J. S., *Evidences of An Inherent Error in Measurement of Total-Head Pressure at Supersonic Speeds*, Aeronautical Engineering Review, Vol. 12, No. 11, p. 47–51, November, 1953.

[2] Sherman, F. S., *New Experiments on Impact-Pressure Interpretation in Supersonic and Subsonic Rarefied Air Streams*, NACA TN 2995, 1953.

The Behavior of the Boundary Layer in the Region of Transition from Laminar to Turbulent Flow*

Jerome Persh
U.S. Naval Ordnance Laboratory, White Oak, Silver Spring, Md.
January 10, 1955

ALTHOUGH A GREAT DEAL of analytical and experimental work has been done on boundary-layer stability and the determination of transition Reynolds Numbers, at the present time little is known of the behavior of the boundary layer in the transition region. As a first step in filling this gap in the available knowledge of boundary-layer characteristics, a large amount of boundary-layer velocity profile and pressure distribution data have been collected from a number of investigations on flat plates, hollow cylinder models, and airfoils in both subsonic and supersonic flows.[1–6] The great bulk of the experimental data collected was obtained with conventional pitot-static tubes; however, it is felt that since gross changes in the internal structure of the boundary layer are reflected in the mean velocity profile the use of these data in a qualitative analysis such as this is justifiable.

In studying the behavior of the boundary-layer parameters derived from all of the experimental data collected, it was noted that near the leading edge of a given surface the values of the boundary-layer shape parameter, defined as

$$H = \frac{\delta^*}{\theta} = \frac{\int_0^\delta \left(1 - \frac{u}{u_\infty}\right) dy}{\int_0^\delta \frac{u}{u_\infty}\left(1 - \frac{u}{u_\infty}\right) dy}$$

assume their characteristic laminar value of about 2.6 (for the zero pressure gradient cases) and at some point along the surface an abrupt drop occurs in the curve showing the variation of H with distance. From this point downstream to the end of the surface, a smooth curve could be used to join the data points. The point where the value of H dropped abruptly is defined as the mean position for the start of transition, and the point where the value of H first reached the characteristic turbulent value (about 1.3–1.4) is then defined as the end of transition. A graphical representation of the flow model described is shown in Fig. 1. In this figure the start of transition is shown as an abrupt drop in the H curve. This is a physically unrealistic situation, and unpublished detailed experiments at the National Bureau of Standards confirm the expectation of a rounding of the curve at this point. Few of the sets of data examined were of sufficient detail to show this rounding of the H curve however, and therefore for the present analysis the abrupt drop in the H curve will be assumed. In any case the values of H drop rapidly just downstream of the transition point, and the mean axial position of the start of transition will be little affected by the roundness of the curve at this point.

Fig. 2 demonstrates the applicability of the flow model to several sets of experimental data with zero pressure gradient which were selected as typical of the many sets of data examined. For each of the cases shown it is apparent that an abrupt drop in the H curve occurs at the end of the laminar zone, and a smooth

* Paper presented at the Division of Fluid Mechanics of the American Physical Society Meeting at Langley Field, Va., on November 24, 1954.

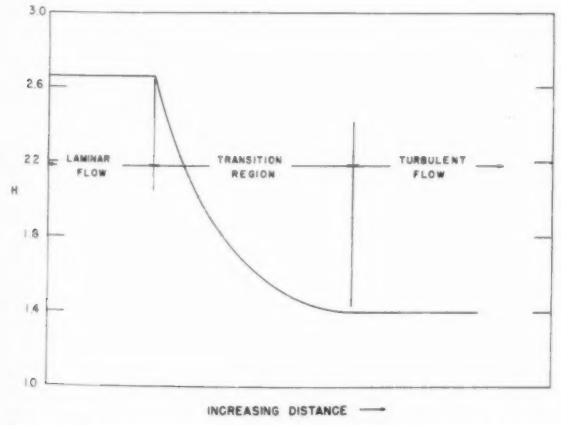


FIG. 1. Shape parameter variation with distance along surface.

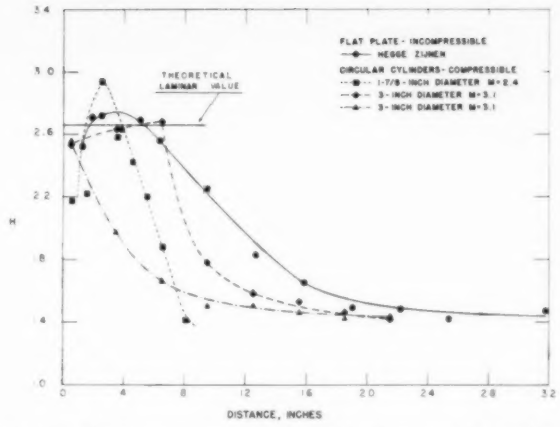

FLAT PLATE - INCOMPRESSIBLE
HEGGE ZIJNEN
CIRCULAR CYLINDERS - COMPRESSIBLE
1-7/8 - INCH DIAMETER M = 2.4
3 - INCH DIAMETER M = 3.1
3 - INCH DIAMETER M = 3.1
THEORETICAL LAMINAR VALUE
H
DISTANCE, INCHES

FIG. 2. Shape parameter variation with distance along surface for zero pressure gradient data.

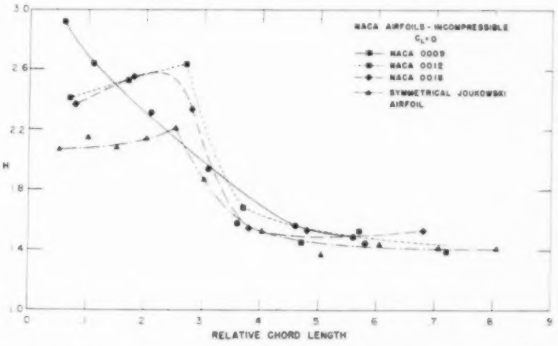


FIG. 3. Shape parameter variation with distance along surface for pressure gradient data.

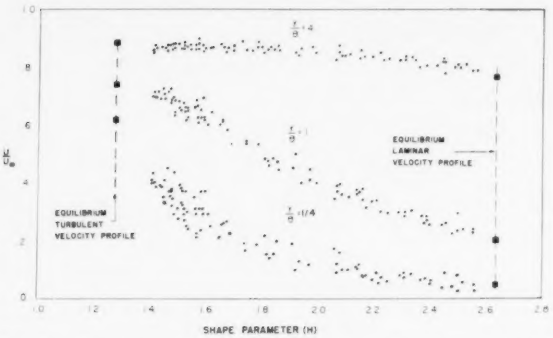


FIG. 4. Correlation of transition region boundary-layer velocity profiles.

curve joins this point and the downstream turbulent region. There appears to be no correlation of the H curves in the transition region. This is an anticipated result because probably the behavior of the boundary layer in this region depends on a combination of parameters. It should be noted that the values of H for the supersonic data were evaluated using incompressible definitions. Because compressible boundary-layer velocity profiles are similar to incompressible profiles, parameters derived from supersonic velocity profiles, using incompressible definitions, have much the same values as the incompressible parameters. Although incompressible values of H for compressible boundary-layer velocity profiles are fictitious values in a sense, their use provides a common basis for comparison between the incompressible and compressible flows. It is important to point out that curves showing the compressible values of H plotted against distance exhibit the same behavior as those shown in Fig. 2 but are displaced in an upward direction by an amount that depends on Mach Number. In Fig. 3 are shown four typical pressure gradient cases. Each of these exhibits the same characteristic drop in the H curve as shown on Fig. 2, with the exception of the laminar region near the leading edge. The large variations in H values in this region are to be expected because the laminar H value depends on the local pressure gradient which was different for each of the cases shown. It is important to note that it is only coincidence that cases were selected for which the start of transition occurred at roughly the same axial location.

To determine whether transition region velocity profiles are of a single parameter family of curves, these velocity profile data were isolated and a correlation similar to that often presented for turbulent boundary-layer velocity profiles was attempted. Values of u/u_∞ were plotted against H for several constant values of y/θ. This is shown in Fig. 4. Because of the fact that no significant trends could be detected between sets of data, no effort was made to identify the points with respect to investigation. Also shown in this figure are the positions of the equilibrium laminar and turbulent velocity profiles in this coordinate system with the corresponding y/θ intercepts denoted as square data points. The results shown in this figure indicate that transition region boundary-layer velocity profiles are of a single parameter family of curves and represent a smooth continuous reversion from laminar to turbulent flow.

REFERENCES

[1] von der Hegge Zijnen, B. G., *Measurements of the Velocity Distribution in the Boundary Layer Along a Plane Surface*, Report 6, Aero. Lab. Tech. H. S., Delft., 1924.

[2] Brinich, P. F., and Diaconis, N. S., *Boundary Layer Development and Skin Friction at Mach Number 3.05*, NACA TN 2742, July, 1952.

[3] O'Donnell, Robert M., *Experimental Investigation at a Mach Number of 2.41 of Average Skin-Friction Coefficients and Velocity Profiles for Laminar and Turbulent Boundary Layers and an Assessment of Probe Effects*, NACA TN 3122, January, 1954.

[4] Silverstein, A., and Becker, J. V., *Determination of Boundary Layer Transition on Three Symmetrical Airfoils in the NACA Full-Scale Wind Tunnel*, NACA Report 637, 1939.

[5] Schubauer, G. B., *Airflow in the Boundary Layer of an Elliptic Cylinder*, NACA Report 652, 1939.

[6] Fage, A., and Falkner, V. M., *An Experimental Determination of the Intensity of Friction on the Surface of an Aerofoil*, Aero. Res. Comm., R. & M. No. 1315 (Ae. 470), April, 1930.

On Compressible Laminar Boundary-Layer Characteristics in a Pressure Gradient with Heat Transfer

Morris Morduchow
Associate Professor of Applied Mathematics, Polytechnic Institute of Brooklyn, Brooklyn, N.Y.
January 3, 1955

FROM AN ANALYSIS[1, 2] of similar solutions of the compressible laminar boundary-layer equations leading to an ordinary differential equation related to the Falkner-Skan equation, Li and Nagamatsu have shown that certain conclusions regarding the effect of pressure gradient and wall temperature on skin friction and separation can be derived. It is inferred, for example, that in a favorable pressure gradient, lowering the wall temperature tends to reduce the skin friction, while in an adverse pressure gradient it tends to delay separation. Moreover, the effect of pressure gradient is enhanced when the wall temperature is increased, or equivalently, the effect of the pressure gradient tends to be diminished by a lowering of the wall temperature.

It may be of interest, in this connection, to also note references 3 and 4, in which a theoretical investigation has been made to determine the effect, in general, of pressure gradient, wall temperature, and Mach Number on the laminar boundary-layer characteristics, particularly on skin friction, heat-transfer coefficient, separation, and stability. Reference 3 (which includes, in addition, the effect of a small normal mass flow at the wall) has been published recently, while reference 4 is to be published shortly. The analysis in references 3 and 4 is based on the Karman-Pohlhausen integral type of method, in conjunction with fourth, sixth, and seventh degree velocity and stagnation enthalpy profiles. General conclusions are derived from the equations thus developed, and numerical examples are given to illustrate the various results in detail. Despite the considerable difference in the methods of analysis used, it is noteworthy that the qualitative results reported in references 1 and 2 are all in essential agreement with those conclusions derived in References 3 and 4 to which they correspond.

REFERENCES

[1] Li, Ting-Yi, and Nagamatsu, Henry T., *Similar Solutions of Compressible Boundary-Layer Equations*, Journal of the Aeronautical Sciences, Readers' Forum, Vol. 20, No. 9, p. 653, September, 1953.

[2] Li, Ting-Yi, and Nagamatsu, Henry T., *The Effect of Heat Transfer on the Behavior of a Compressible Laminar Boundary Layer Under Favorable and Adverse Pressure Gradients*, Journal of the Aeronautical Sciences, Readers' Forum, Vol. 21, No. 12, p. 850, December, 1954.

[3] Morduchow, Morris, and Galowin, Lawrence, *The Compressible Laminar Boundary Layer in a Pressure Gradient over a Surface Cooled by Fluid Injection*, Proc. of the Iowa Thermodynamics Symp., April 27–28, 1953, State University of Iowa, Iowa City, Iowa.

[4] Morduchow, Morris, and Grape, Richard G., *On Separation, Stability and Other Properties of the Compressible Laminar Boundary Layer with Pressure Gradient and Heat Transfer*, PIBAL Rep. No. 237, January, 1954. NACA TN 3296, 1955.